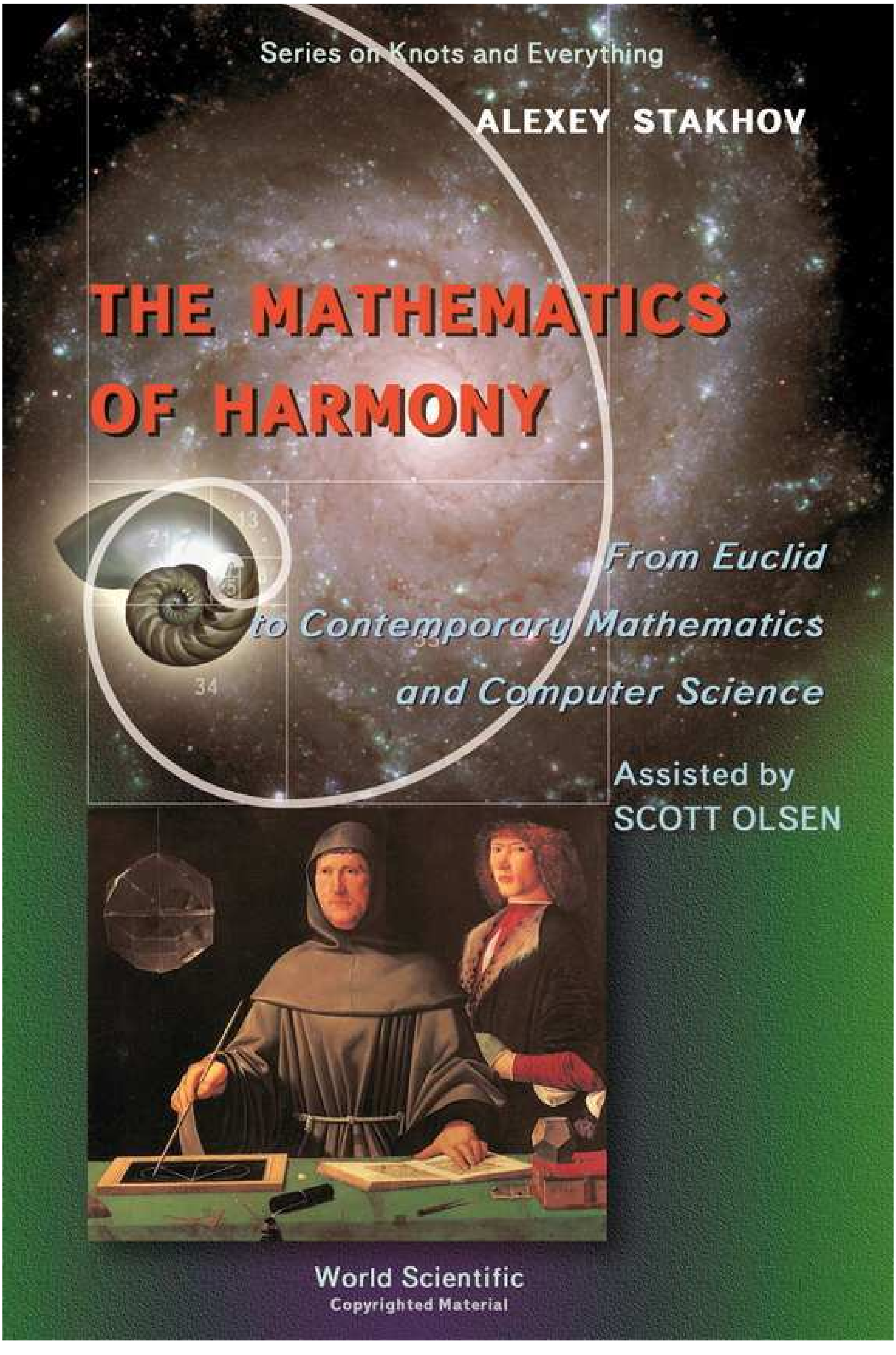
Alexey Stakhov
Mathematics of Harmony: From Euclid to Contemporary Mathematics and Computer Science



Alexey Stakhov
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Mathematics of Harmony: From Euclid to Contemporary Mathematics and Computer Science

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Abstract

This volume is a result of the author's four decades of research in the field of Fibonacci numbers and the Golden Section and their applications. It provides a broad introduction to the fascinating and beautiful subject of the "Mathematics of Harmony," a new interdisciplinary direction of modern science. This direction has its origins in "The Elements" of Euclid and leads to many unexpected applications in contemporary mathematics (a new approach to a history of mathematics, the generalized Fibonacci and Lucas numbers and the generalized golden proportions, the "golden" algebraic equations, the generalized Binet formulas, Fibonacci and "golden" matrices), theoretical physics (hyperbolic Fibonacci and Lucas functions and new hyperbolic models of Nature), computer science (algorithmic measurement theory, number systems with irrational radices, Fibonacci computers, ternary mirror-symmetrical arithmetic, a new theory of coding and cryptography based on the Fibonacci and "golden" matrices), and mathematical education. The Harmony Mathematics can be considered as an alternative way of the mathematics development.

The book is intended for a wide audience including mathematics teachers of high schools, students of colleges and universities and scientists in the field of mathematics, theoretical physics and computer science. The book may be used as an advanced textbook by graduate students and even ambitious undergraduates in mathematics and computer science. Also all researchers interested in the golden section and Fibonacci numbers are the readers of the book.

Readership: Researchers, teachers and students in mathematics (especially those interested in the Golden Section and Fibonacci numbers), theoretical physics and computer science.

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Academician Mitropolsky's Commentary on the Scientific Research of Ukrainian Scientist Professor Alexey Stakhov, Doctor of Sciences in Computer Science

I have followed the scientific career of Professor Stakhov for a long time—seemingly since the publication of his first book, *Introduction into Algorithmic Measurement Theory* (1977), which was presented by Professor Stakhov in 1979 at the scientific seminar of the Mathematics Institute of the Ukrainian Academy of Sciences. I became especially interested in Stakhov's scientific research after listening to his brilliant speech at a session of the Presidium of the Ukrainian Academy of Sciences in 1989. In his speech, Professor Stakhov reported on scientific and engineering developments in the field of "Fibonacci computers" that were conducted under his scientific supervision at Vinnitsa Technical University.

I am very familiar with Stakhov's scientific works—because at my recommendation—many of his papers were published in various Ukrainian academic journals. In April 1998, I invited Professor Stakhov to lecture on his scientific research at a meeting of the Ukrainian Mathematical Society. His lecture produced a positive reaction from the members of the society. At the request of Professor Stakhov, I wrote the introduction to his book, *Hyperbolic Fibonacci and Lucas Functions*, which was published in 2003 in small edition. In recent years, I have been actively corresponding with Professor Stakhov, and we have discussed many new scientific ideas. During these discussions I became very impressed with his qualifications and extensive knowledge in regards to his research in various areas of modern science. In particular, I am surprised by his knowledge in the field of mathematics history.

The basic feature of Stakhov's scientific creativity consists of his unconventional outlook upon ancient mathematical problems. For example, I shall begin with my review of his book *Introduction into Algorithmic Measurement Theory* (1977). This publication rewarded Professor Stakhov with recognition in the field of modern theoretical metrology. In this book, Professor Stakhov introduced a new mathematical direction in measurement theory—the Algorithmic Measurement Theory.

In 1993, I recommended a publication of an original paper, prepared by Professor Alexey Stakhov and Ivan Tkachenko, entitled "Fibonacci Hyperbolic Trigonometry," for publication in the journal *Reports of the Ukrainian Academy of Sciences.* The paper addressed a new theory of hyperbolic Fibonacci and Lucas functions. This paper demonstrated the originality of Stakhov's scientific thinking. In fact, the classical hyperbolic functions were widely known and were used as a basis of Non-Euclidean geometry developed by Nikolay Lobachevsky. It is quite peculiar that at the end of 20th century the Ukrainian scientists Stakhov and Tkachenko discovered a new class of the hyperbolic functions based on the Golden Section, Fibonacci and Lucas numbers that has "strategic" importance for the development of modern mathematics and theoretical physics.

In 1999, I also recommended Stakhov's article "A Generalization of the Fibonacci Q-Matrix"—which was presented by the author in English—to be published in the journal *Reports of the Ukrainian Academy of Sciences* (1999, Vol. 9). In this article, Professor Stakhov generalized and developed a new theory of the Q-matrix which had been introduced by the American mathematician Verner Hoggatt—a founder of the Fibonacci-Association. Stakhov introduced a concept of the Q_p-matrixes (p=0, 1, 2, 3...), which are a new class of square matrixes (a number of such matrixes is infinite). These matrixes are based on so-called Fibonacci p-numbers, which had been discovered by Stakhov while investigating "diagonal sums" of the Pascal triangle. Stakhov discovered a number of rather unusual properties of the Q_p-matrixes. In particular, he proved that the determinant of the Q_p-matrix or any power of that matrix is equal to +1 or -1. It is my firm belief that a theory of Q_p-matrixes can be regarded as a new fundamental result in the classic matrix theory.

In 2004, *The Ukrainian Mathematical Journal* (Vol. 8) published Stakhov's article “The Generalized Golden Sections and a New Approach to Geometrical Definition of Number.” In this article, Professor Stakhov obtained mathematical results in number theory. The following are worth mentioning:

1. A Generalization of the Golden Section Problem. The essence of this generalization is extremely simple. Let us set a non-negative integer (p=0, 1, 2, 3, ...) and divide a line segment AB at the point C in the following proportion:

$$\frac{CB}{AC}=\left(\frac{AB}{CB}\right)^p$$

We then get the following algebraic equation:

$$x^{p+1} = x^p + 1.$$

The positive roots of this algebraic equation were named the *Generalized Golden Proportions* or the *Golden p-proportions* τ_p. Let's ponder upon this result. Within several millennia, beginning from Pythagoras and Plato, mankind has widely used the known classical Golden Proportion as some unique number, and at the end of the 20th century, the Ukrainian scientist Stakhov had generalized this result and proved the existence of the infinite number of the Golden Proportions, that have the same rights as the classical Golden Proportion to express Harmony. Moreover, Stakhov proved that the golden p-proportions τ_p ($1\le\tau_p\le2$) represented a new class of irrational numbers, which express some unknown mathematical properties of the Pascal triangle. Clearly, such mathematical result is of fundamental importance for the development of modern science and mathematics.

2. Codes of the Golden *p*-proportions. Using a concept of the golden p-proportion, Stakhov introduced a new definition of real number in the form:

$$A=\sum_i a_i\tau_p^i \quad (a_i\in\{0,1\})$$

He named this sum the “Code of the golden p-proportion.” Stakhov proved that this concept, which is an elaboration of the well known “Newton’s definition” of real numbers, can be used for the creation of a new theory for real numbers. Furthermore, he proved that this result could be used for the creation of new computer arithmetic and new computers—*Fibonacci computers.* Stakhov not only introduced the idea of Fibonacci computers, but he also organized the engineering projects on the creation of such computer prototypes in the Vinnitsa Polytechnic Institute from 1977-1995. Sixty five foreign patents for inventions in the field of Fibonacci computers have been issued by the state patent offices of the United States, Japan, England, France, Germany, Canada, and other countries; these patents have confirmed the significance of Ukrainian science—and of Professor Stakhov—in this important computer area.

In recent years, the area of Professor Stakhov’s scientific interests has moved more and more towards the area of mathematics. For example, his lecture “The Golden Section and Modern Harmony Mathematics” delivered at the Seventh International Conference on Fibonacci Numbers and their Applications in Graz, Austria in 1996, and then repeated in 1998 at the Ukrainian Mathematical Society, confirmed a new trend in Stakhov’s scientific research. This lecture was impressive and it created much discussion on Stakhov’s new research.

Currently, Professor Stakhov is an actively working scientist who publishes his scientific papers in many internationally recognized journals. Most recently, he has published many fundamental papers in the international journals: *Computers & Mathematics with Applications*; *The Computer Journal*; *Chaos, Solitons & Fractals*; *Visual Mathematics*; and others. This fact demonstrates, undoubtedly, tremendous success not only for Professor Stakhov, but also for Ukrainian science.

Stakhov’s articles finish a cycle of his long-term research on the creation of a new direction in mathematics: **Mathematics of Harmony**. One may wonder what place in the general theory of mathematics this work may have. It seems to me—that in the last few centuries—as Nikolay Lobachevsky said, “Mathematicians have turned all their attention to the advanced parts of analytics, and have neglected the origins of Mathematics and are not willing to dig the field that has already been harvested by them and left behind.” As a result, this has created a gap between “Elementary Mathematics”—the basis of modern mathematical education—and “Advanced Mathematics.” In my opinion, the Mathematics of Harmony developed by Professor Stakhov fills that gap. **Mathematics of Harmony** is a huge theoretical contribution to the development of “Elementary Mathematics,” and as such should be considered of great importance for mathematical education.

It is imperative to mention that Professor Stakhov focuses his organizational work on stimulating research in the field of theory surrounding Fibonacci numbers and the Golden Section; he also assists in spreading knowledge among broad audiences inside the scientific community. In 2003, under Professor Stakhov’s initiative and scientific supervision, the international conference on “Problems of Harmony, Symmetry, and the Golden Section in Nature, Science, and Art” was held. At this conference, Professor Stakhov was elected President of the International Club of the Golden Section, confirming his official status as leader of a new scientific direction that is actively progressing modern science.

Professor Stakhov proposed the discipline “Mathematics of Harmony and the Golden Section” for the mathematical faculties of pedagogical universities. In essence, this mathematical discipline can be considered the beginning of mathematical education reform—which is based on the principles of Harmony and the Golden Section. It should be noted that such discipline was delivered by Professor Stakhov during 2001-2002 for the students of Faculty of Physics and Mathematics at Vinnitsa State Pedagogical University. I have no doubts surrounding the usefulness of such discipline for future teachers in mathematics and physics. I believe that Professor Stakhov has the potential to write a textbook on this discipline for pedagogical universities, and also a textbook on *Mathematics of the Golden Section* for secondary schools.

I do not doubt that “Mathematics of Harmony,” created by Professor Stakhov, has huge interdisciplinary importance as this mathematical discipline touches the bases of many sciences—including: mathematics, theoretical physics, and computer science. Stakhov suggested mathematical education reform on the basis of the ideas of Harmony and the Golden Section. This reform opens the doors for the development of mathematical and general education curriculum. This reform will greatly contribute to the development of the new scientific outlook based on the principles of Harmony and the Golden Section.

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1. "Strategic Mistakes" in the Mathematics Development

1.1. *A Severance of the Relations between Mathematics and Theoretical Natural Sciences*

The book *Mathematics. The Loss of Certainty* [6] by Morris Kline, Professor Emeritus of Mathematics Courant Institute of Mathematical Sciences of New York University, is devoted to the analysis of the crisis of the 20th century mathematics.

Kline wrote: "*The history of mathematics is crowned with glorious achievements but also a record of calamities. The loss of truth is certainly a tragedy of the first magnitude, for truths are man's dearest possessions and a loss of even one is cause for grief. The realization that the splendid showcase of human reasoning exhibits a by no means perfect structure but one marred* *by shortcomings and vulnerable to the discovery of disastrous contradiction at any time is another blow to the stature of mathematics. But there are not the only grounds for distress. Grave misgivings and cause for dissension among mathematicians stem from the direction which research of the past one hundred years has taken. Most mathematicians have withdrawn from the world to concentrate on problems generated within mathematics. They have abandoned science. This change in direction is often described as the turn to pure as opposed to applied mathematics."*

Further we read: *"Science had been the life blood and sustenance of mathematics. Mathematicians were willing partners with physicists, astronomers, chemists, and engineers in the scientific enterprise. In fact, during the 17th and 18th centuries and most of the 19th, the distinction between mathematics and theoretical science was rarely noted. And many of the leading mathematicians did far greater work in astronomy, mechanics, hydrodynamics, electricity, magnetism, and elasticity than they did in mathematics proper. Mathematics was simultaneously the queen and the handmaiden of the sciences."*

Kline notes that our great predecessors did not be interested in problems of the "pure mathematics" placed in the forefront of the 20th century mathematics. In this connection, Kline writes: *"However, pure mathematics totally unrelated to science was not the main concern. It was a hobby, a diversion from the far more vital and intriguing problems posed by the sciences. Though Fermat was the founder of the theory of numbers, he devoted most of his efforts to the creation of analytic geometry, to problems of the calculus, and to optics ... He tried to interest Pascal and Huygens in the theory of numbers but failed. Very few men of the 17th century took any interest in that subject."*

Felix Klein, who was the recognized head of the mathematical world, considered necessary to make a protest against striving for abstract, "pure" mathematics: *"We cannot help feeling that in the rapid developments of modern thought, our science is in danger of becoming more and more isolated. The intimate mutual relation between mathematics and theoretical natural science which, to the lasting benefit of both sides, existed ever since the rise of modern analysis, threatens to be disrupted."*

Richard Courant, who headed the Institute of Mathematical Sciences of New York University, also treated disapprobatory the passion to the "pure" mathematics. He wrote in 1939: *"A serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from the definition and postulates that must be consistent but otherwise may be created by the free will of mathematicians. If this description were accurate, mathematics could not attract any intelligent person. It would be a game with definitions, rules, and syllogisms without motivation or goal. The notion that the intellect can create meaningful postulational system at its whim is a deceptive half-truth. Only under the discipline of responsibility to the organic whole, only guided by intrinsic necessity, can be free mind achieve results of scientific value."*

At present, mathematicians turned their attention to the old mathematical problems formulated by the Great mathematicians of the past. *Fermat's Last Theorem* is one of them. This theorem can be formulated very simply. Let us prove that for $n>2$ any integers x, y, z do not satisfy the correlation $x^n + y^n = z^n$. The theorem was formulated by Fermat in 1637 on the margins of Diofant's book *Arithmetics* together with the postscript that the witty proof found by him is too long that to be placed here. As is well known, many outstanding mathematicians (Euler, Dirichlet, Legandre and others) tried to solve this problem. The proof of Fermat's Last Theorem was completed in 1993 by Andrew Wiles, a British mathematician working at Princeton in the USA. The proof stated on 130 pages was published in *Annals of Mathematics*.

As is well known, *Gauss* is a recognized specialist in number theory what is confirmed by the publication of his book *Arithmetical Researchers* (1801). In this connection, it is curiously to know Gauss' opinion about *Fermat's Last Theorem*. Gauss explained in one of his letters why he did not study Fermat's problem. From his point of view**, "the Fermat hypothesis was an isolated theorem and so had little interest" [6].** Gauss **"did hope that perhaps the Fermat**

hypothesis might be proven on the basis of other work he had done, but it would be one of the less interesting corollaries" [6]. Without doubts, Gauss' opinion about *Fermat's Last Theorem* belittles somewhat Andrew Wiles' proof of this theorem. And after Gauss, we can ask the following questions:
(1) What significance has the Fermat Last Theorem for the development of modern science?
(2) Can we compare the proof of the Fermat hypothesis with the discovery of the Non-Euclidean geometry in the first half of the 19th century and with other outstanding mathematical discoveries?
(3) Whether is a proof of the Fermat Last Theorem only an "aimless play of intellect" and the demonstration of the human intellect power - and not more?

Thus, after Felix Klein, Richard Courant and other famous mathematicians, Morris Kline asserts that **the main reason of the contemporary crisis of mathematics is the severance of the relations between mathematics and theoretical natural sciences what is the greatest "strategic mistake" of the 20th century mathematics.**

1.2. *The Neglect of the "Beginnings"*

The Russian Great mathematician Kolmogorov wrote the Preface to the Russian translation of Lebegue's book *About the Measurement of Magnitudes* [3]. He wrote here that *"there is a tendency among mathematicians to be ashamed of the mathematics origin. In comparison with crystalline clearness of the theory development, since its basic notions and assumptions, it seems unsavory and unpleasant pastime to rummage in the origin of these basic notions and assumptions. All building of the school algebra and all mathematical analysis might be constructed on the notion of real number without the mention about the measurement of specific magnitudes (lengths, areas, time intervals, and so on). Therefore, one and the same tendency shows itself on different stages of education and with different degree of courage to introduce numbers as possibly sooner and further to speak only about numbers and correlations among them. Lebegue protests against this tendency!"*

In this statement, Kolmogorov noticed one peculiarity of mathematicians – a diffident relation to the "beginnings" of mathematics, by other words, the neglect of the "beginnings" ("on different stages of education and with different degree of courage"). However, long before Kolmogorov, Nikolay Lobachevski paid attention to this tendency (see an epigraph to this book).

However, just Lobachevski had demonstrated by his researches that the "beginnings" of mathematical sciences, in particular, Euclid's *Elements* are inexhaustible source of new mathematical ideas and discoveries. Lobachevski's *Geometric Researches on Parallel Lines* (1840) begins by the following words: *"I have found in geometry some imperfections, which are reasons of the fact why this science did not overstep until now the limits of Euclid's Elements. We are talking here about the first notions about geometric magnitudes, about the measurement methods and, at last, about the important gap in the theory of parallel lines ..."*

As is well known, Lobachevski, in contrast to other mathematicians did not neglect the "beginnings." The thorough analysis of the *Fifth Euclidean Postulate* ("the important gap in the theory of parallel lines") had led him to the creation of the Non-Euclidean geometry - the most important mathematical discovery of the 19th century.

1.3. *The Neglect of the Golden Section*

Pythagoreans had put forward for the first time a brilliant idea about harmonic structure of the Universe including not only nature and person but also all cosmos. According to Pythagoreans, a harmony is inner connection of things without which cosmos cannot exist. At last, according to Pythagoras, harmony has numerical expression, that is, it is connected with number concept. Aristotle noticed in his Metaphysics just this peculiarity of the Pythagorean doctrine: *"The so-called Pythagoreans, studying mathematical sciences, for the first time have moved them* *forward and, basing on them, began to consider mathematics as the beginnings of all things... Because all things became similar to numbers, and numbers occupied the first place in all nature, they had assumed that the elements of numbers are the beginning of all things and that all universe is harmony and number."*

Pythagoreans recognized that the shape of the Universe should be harmonious and all "elements" of the Universe are connected with harmonious figures. Pythagoras taught that the Earth came into being from the cube, the Fire from the pyramid (the tetrahedron), the Air from the octahedron, the Water from the icosahedron, the sphere of the Universe (the ether) from the dodecahedron.

The famous Pythagorean doctrine about the "harmony of spheres" is connected with the harmony concept. Pythagoras and his followers considered that the movement of heavenly bodies around the central world fire creates wonderful music, which is perceived not by ear, but by intellect. The doctrine about the "harmony of spheres," about the unity of micro and macro cosmos, the doctrine about proportions - all in the whole build up the base of the Pythagorean doctrine.

The main conclusion, which follows from Pythagorean doctrine, consists in the fact that harmony is objective; it exists independently from our consciousness and is expressed in harmonious structure of the Universe since cosmos up to microcosm. However, if harmony is objective, it should become a subject of mathematical researches.

Pythagorean doctrine about numerical harmony of the Universe had influenced on the development of all subsequent doctrines about nature and essence of harmony. This brilliant doctrine got reflection and development in the works of the Great thinkers, in particular, in Plato's cosmology. In his works, Plato develops Pythagorean doctrine and emphasizes especially a cosmic significance of harmony. He is firmly convinced that harmony can be expressed by numerical proportions. The Pythagorean influence is traced especially in his *Timeous*, where Plato, after Pythagoras, develops a doctrine about proportions and analyzes a role of the regular polyhedrons (Platonic Solids), which, in his opinion, underlie the Universe.

The "golden section," which was called in that period the "division in the extreme and mean ratio," played especial role in ancient science, including Plato's cosmology. Aexey Losev, the Russian brilliant philosopher and researcher of the aesthetics of Ancient Greece and Renaissance, had expressed his relation to the "golden section" and Plato's cosmology in the following words: *"From Plato's point of view, and generally from the point of view of all antique cosmology, the universe is a certain proportional whole that is subordinated to the law of harmonious division, the Golden Section... Their system of cosmic proportions is considered sometimes in literature as curious result of unrestrained and preposterous fantasy. Full anti-scientific helplessness sounds in the explanations of those who declare this. However, we can understand the given historical and aesthetical phenomenon only in the connection with integral comprehension of history, that is, by using dialectical-materialistic idea of culture and by searching the answer in peculiarities of the ancient social existence."*

We can ask the question: in what way the "golden section" is reflected in contemporary mathematics? Unfortunately, we can give the following answer: in no way. In mathematics, Pythagoras and Plato's ideas are considered as "curious result of unrestrained and preposterous fantasy." Therefore, a majority of mathematicians consider a study of the "golden section" as a pastime, which is unworthy for serious mathematicians. Unfortunately, we can find the neglect of the "golden section" in contemporary theoretical physics. In 2006 the Publishing House "BIMON" (Moscow) had published the interesting scientific book "Metaphysics. Century XXI" [85]. In the Preface to the book, the compiler and editor of the book Professor Vladimirov (Moscow University) wrote: *"The third part of the book is devoted to the discussion of the numerous examples of manifestation of the "golden section" in art, biology and in the world surrounding us. However, as it is no paradoxical, the "golden proportion" in contemporary theoretical physics is reflected in no way. To be convinced in this fact, it is enough to browse 10* *volumes of Theoretical Physics by Landau and Lifshitz. A time came to fill this gap in physics, all the more that the "golden proportion" is connected intimately with metaphysics and trinitarity."*

In this connection, we should remember Kepler's well-known saying concerning the "golden section": "*Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first, we may compare to a measure of gold; the second we may name a precious stone."*

Many mathematicians consider Kepler's saying as big overstatement for the "golden section." However, we should not forget that Kepler was both brilliant astronomer, and also Great physicist and Great mathematician (in contrast to the mathematicians who criticizes him). Kepler was one of the first scientists, who raised a problem of the study of the "Harmony of the Universe" in his book *Harmonices Mundi* ("Harmony of the World"). In *Harmony Mundi*, he attempted to explain the proportions of the natural world - particularly the astronomical and astrological aspects - in the terms of music (*Music of the Spheres*). Kepler began to explore *Regular Polygons* and *Regular Solids*, including the figures known as *Kepler's Solids*. From there, he extended his harmonic analysis to music, meteorology and astrology; harmony resulted from the tones made by the souls of heavenly bodies - and in the case of astrology, the interaction between those tones and human souls. In the final portion of the work (Book V), Kepler dealt with planetary motions, especially relationships between orbital velocity and orbital distance from the Sun. Similar relationships had been used by other astronomers, but Kepler - with Tycho's data and his own astronomical theories - treated them much more precisely and attached new physical significance to them.

Thus, the neglect of the "golden section" and the "harmony idea" is one more "strategic mistake" not only mathematics but also theoretical physics.

This mistake originated a number of other "strategic mistakes" in the mathematics development.

1.4. ***The One-sided Interpretation of Euclid's Elements***

As is well-known, Euclid's *Elements* is the main mathematical work of the Greek science. This work is devoted to axiomatic construction of geometry. Such look on the *Elements* is the most widespread in contemporary mathematics. However, there is another point of view on the *Elements* suggested by **Proclus Diadoch** (412-485), the best commentator of Euclid's *Elements.* As is well-known, the Book XIII, that is, the final book of Euclid's *Elements,* is devoted to the description of the theory of the 5 Regular Polyhedrons that played a predominate role in Plato's cosmology. They are well known in modern science under the name *Platonic Solids.* Proclus attracted a particular attention to this fact. As Soroko emphasizes [25], in Proclus opinion, Euclid *"created the Elements supposedly not with purpose to present axiomatic approach to geometry, but in order to give a systematic theory of the construction of the 5 Platonic Solids, in passing by lighting some most important achievements of mathematics."* Thus, "Proclus' hypothesis" allows to suppose that the *Pythagorean Doctrine about Numerical Harmony of Universe* well-known in the ancient science and *Plato's Cosmology* based on the regular polyhedrons were embodied in Euclid's *Elements,* the greatest mathematical work of the Greek science. From this point of view, **we can interpret Euclid's *Elements* as the first attempt to create "Mathematical Theory of Harmony" what was the main idea of the Greek science.**

This hypothesis is confirmed by geometric theorems of Euclid's *Elements.* **A problem of division in extreme and mean ratio** described in Theorem II.11 is one of them. This division, which was named later the "golden section," was used by Euclid for geometric construction of the isosceles triangle with the angles 72°, 72° и 36° (the "golden" isosceles triangle) and then of regular pentagon and dodecahedron. We should tell with great regret that "Proclus' hypothesis" did not be perceived by modern mathematicians who continue to consider axiomatic construction of geometry the main achievement and the main purpose of Euclid's *Elements.*

The one-sided interpretation of Euclid's *Elements* is one more "strategic mistake" in the mathematics development. This "strategic mistake" resulted in the distorted picture of the history of mathematics.

1.5. ***The One-sided Approach to the Mathematics Origin***

As is well known, a traditional approach to the mathematics origin consists in the following [1]. Historically, two practical problems stimulated the mathematics development on the earlier stages of its development. We are talking about the **"count problem"** and the **"measurement problem."** The "count problem" resulted in the creation of the first methods of number representation and the first rules for the fulfillment of arithmetical operations (Babylonian sexagecimal number system, Egyptian decimal arithmetic and so on). The formation of the concept of **natural number** was the main result of this long period in the mathematics history. On the other hand, the "measurement problem" underlies the geometry creation ("Measurement of the Earth"). A discovery of the **incommensurable line segments** is considered the major mathematical discovery in this field. This discovery resulted in the introduction of **irrational numbers,** the next fundamental mathematical concept after natural numbers**.**

The concepts of **natural number** and **irrational number** are the major mathematical concepts, without which it is impossible to imagine the existence of mathematics. These two fundamental concepts underlie "classical mathematics."

Unfortunately, mathematicians neglected the "harmony problem" and the "golden section." These important concepts had influenced the mathematics development. As result, we have the one-sided look on the mathematics origin what is one more "strategic mistake" in the mathematics development.

1.6. ***The Greatest Mathematical Mystification of the 19th Century***

The "strategic mistake" influenced considerably on the mathematics development was made in the 19th century. We are talking about **Cantor's Theory of Infinite Sets.** This theory had brought mathematics a number of useful mathematical results and was used in the "golden section theory" [142]. However, Cantor's theory was perceived by the 19th century mathematicians without critical analysis. The end of the 19th century was a culmination point in recognizing Cantor's theory of infinite sets. The official proclamation of the set-theoretical ideas as the mathematics base was made in 1897: this statement was contained in Hadamard's speech on the First International Congress of Mathematicians in Zurich (1897). In his lecture the Great mathematician Hadamard did emphasize that the main attractive reason of Cantor's set theory consists in the fact that for the first time in mathematics history the classification of the sets was made on the base of a new concept of "cardinality" and the amazing mathematical outcomes inspired mathematicians on new and surprising discoveries.

However, very soon the "mathematical paradise" based on Cantor's set theory was destroyed. A finding of paradoxes in Cantor's theory of infinite sets resulted in the crisis in mathematics foundations and this fact cooled interest of mathematicians in Cantor's theory. Recently the Russian mathematician Alexander Zenkin finished critical analysis of Cantor's theory and the introduced by him concept of **"actual infinity,"** which is the main philosophical idea of Cantor's theory.

After the thorough analysis of Cantor's continuum theorem, Alexander Zenkin came to the following unusual conclusion [168]:

1. Cantor's proof of this theorem is not mathematical proof in Hilbert's sense and in the sense of classical mathematics.
2. Cantor's conclusion about non-denumerability of continuum is a "jump" through a potentially infinite stage, that is, Cantor's reasoning contains the fatal logic error of "unproved basis" (a jump to the "wishful conclusion").
3. Cantor's theorem, actually, proves, strictly mathematically, the potential, that is, not finished character of the infinity of the set of all "real numbers," that is, Cantor proves strictly mathematically the fundamental principle of classical logic and mathematics: *"Infinitum Actu Non Datur"* (Aristotle).

Thus, Cantor's theory of infinite sets based on the concept of "actual infinity" contains "fatal logic error" and cannot be the base for mathematics. Its acceptance as mathematics foundation, without proper critical analysis, is one more "strategic mistake" in the mathematics development; Cantor's theory is one of the major reasons of the contemporary crisis in mathematics foundations.

1.7. *The Underestimation of the Binet Formulas*

In the 19th century a theory of the "golden section" was supplemented by one important result. We are talking about the so-called **Binet formulas** for Fibonacci and Lucas numbers given by (2.67) and (2.68). Unfortunately, in the classical mathematics the Binet formulas did not get a proper recognition as, for example, "Euler formulas." Apparently, such attitude to Binet formulas is connected with the "golden mean," which always provoked the "allergy" of mathematicians.

However, the main "strategic mistake" in the underestimation of the Binet formulas consists in the fact that mathematicians could not see in the Binet formulas the beginning of a new class of hyperbolic functions - the hyperbolic Fibonacci and Lucas functions. Such functions were discovered 100 years later by the Ukrainian researchers Bodnar [37], Stakhov, Tkachenko, and Rozin [98, 106, 116, 118]. If the hyperbolic functions Fibonacci and Lucas would be discovered in the 19th century, the hyperbolic geometry and its applications to theoretical physics could get a new impulse in their development.

1.8. *The Underestimation of Felix Klein's Idea Concerning Regular Icosahedron*

In the 19th century the Great mathematician Felix Klein tried to unite all branches of mathematics on the base of the regular icosahedron dual to the dodecahedron [58]. Klein interprets the regular icosahedron based on the "golden section" as a geometric object, which connects the 5 mathematical theories: *Geometry, Galois Theory, Group Theory, Invariant Theory, Differential Equations.* Klein's main idea is extremely simple: *"Each unique geometric object is connected one way or another with the properties of the regular icosahedron."* **Unfortunately, this remarkable idea did not get the development in contemporary mathematics what is one more "strategic mistake" in the mathematics development.**

1.9. *The Underestimation of Bergman's Discovery*

One "strange" tradition exists in mathematics. It is usually for mathematicians to underestimate mathematical achievements of their contemporaries. The epochal mathematical discoveries, as a rule, in the beginning could not be perceived by mathematicians. Sometimes they are subjected to sharp criticism and even to venomous gibes. Only after approximately 50 years, as a rule, after the death of the authors of the outstanding mathematical discoveries, new mathematical theories are recognized and take a worth place in mathematics. The dramatic destinies of Lobachevski, Abel, Galois are known very well in order to describe them more in detail.

In 1957 the American mathematician George Bergman published the article *A Number System with an Irrational Base* [86]. In this article the author developed very unusual extension of the notion of positional number system. He suggested to use the "golden mean" $\tau=\left(1+\sqrt{5}\right)/2$ as a radix of a special number system. If we use the number sequences τ^i {i=0, ±1, ±2, ±3, …} as "digit weights" of the "binary" number system, we get the "binary" number system with an irrational base given by (9.1).

Unfortunately, Bergman's article [86] did not be noticed in that period by mathematicians. Only journalists were surprised by the fact that George Bergman made his mathematical discovery in the age of 12 years! In this connection, the Magazine "TIMES" had published the article about mathematical talent of America. Why the 20th century mathematicians did neglect Bergman's discovery? A cause consisted in the fact that the number systems never was considered a subject of researches for serious mathematician! In 50 years, according to "mathematical tradition," a time had come to evaluate a role of Bergman's system for the development of contemporary mathematics.

In [24] the so-called **"codes of the golden *p*-proportions"** were introduced. They are positional "binary" number systems similar to Bergman's system. However, the "golden *p*-proportions" - positive roots of the algebraic equations $x^{p+1} = x^p + 1$ (p = 0, 1, 2, 3, ...) - are their bases. The "codes of the golden *p*-proportions" are a wide generalization of Bergman's number system (p=1). They originate a new, unknown until now class of positional number systems - **number systems with irrational bases.**

The "strategic" importance of Bergman's system and its generalization - the "codes of the golden *p*-proportion" - consists in the fact that **they overturn our ideas about positional number systems, moreover, our ideas about correlations between rational and irrational numbers.**

As is well known, historically natural numbers were first introduced, after them rational numbers as ratios of natural numbers, and later - after the discovery of the "incommensurable line segments" - irrational numbers, which cannot be expressed as ratios of natural numbers. By using the traditional positional number systems (binary, ternary, decimal and so on), we can represent any natural, real or irrational number through the base of number system (2, 3, 10 and so on). The bases of Bergman's system [86] and "codes of the golden *p*-proportion" [24] are some irrational numbers - the "golden mean" or the golden *p*-proportion. By using these irrational numbers, we can represent natural, real and irrational numbers by using some irrational numbers - golden p-proportions. It is clear that Bergman's system and codes of the golden *p*-proportion can be considered as a **new definition of real number:** such approach has important consequence for number theory.

"Strategic mistake" of the 20th century mathematicians was that they took no notice Bergman's mathematical discovery, which can be considered as the major mathematical discovery in the field of the positional number systems (after the Babylonians discovered the positional principle of number representation).

2. The Classical Mathematics and the Mathematics of Harmony

2.1. *The Classical Mathematics*

It is curiously to note that the Euclidean *Elements* gave an origin of two fundamental branches of mathematics - the *Classical Mathematics* and the *Mathematics of Harmony*. Indeed, all fundamental theories of the classical mathematics - *geometry*, *number theory*, and *measurement* *theory* - had originated from the Euclidean *Elements.* On the other hand, the Euclidean *Elements* gave an origin of the "golden section" ("the division in extreme and mean ratio") and a theory of Platonic Solids what is the beginning of the mathematics of harmony.

Since the ancient time, the classical mathematics and the mathematics of harmony were developing independently one from another. The golden age of the classical mathematics had started in the 17th century and had reached its peak in the 18-th century. In the 17-th century new needs of natural sciences and engineering had compelled mathematicians to concentrate the attention on the development of the mathematical apparatus describing processes of movement. A concept of a *Function* was placed in the forefront and began to play in the mathematics the same role as concept of magnitude and a number in the ancient mathematics. A creation of the infinitesimal analysis, first of all in the form of differential and integral calculus, became a top of this important period in the development of mathematics (17-18th centuries). The new stage in the development of mathematics is integrally connected with the creation in the 17-th century of the mathematical natural sciences, which put forward the explanation of separate natural phenomena by the action of general nature laws formulated mathematically as the main object of mathematics. The most typical feature of this period of the mathematics development is a surprising harmony and close co-operation between mathematics and theoretical natural sciences. Many great mathematicians of the past stood in the center of mathematical natural sciences. The creation of new mathematical models of the natural phenomena was the main task of their mathematical researches.

However, the creation of Cantor's theory of infinite sets based on the idea of "actual infinity" resulted in the crisis in the bases of classical mathematics, which is not overcome till now [167, 168]. A majority of modern mathematicians has departed from mathematical natural sciences and has concentrated the efforts on the problems of "pure mathematics," that is, mathematics for the sake of mathematics. Appreciating applied value of the modern "pure mathematics" for the development of theoretical natural sciences, Kline noted [6]: "*It is extremely unlikely, if one may judge by what happened in the past, that most of the modern mathematical research will ever contribute to the advancement of science; mathematics may be doomed to grope in darkness."* In this connection, the representatives of other sciences were forced to start to create independently new mathematical apparatus necessary for them. Kline noted [6]: "*A few discerning mathematicians have noted that the Newtons, Laplaces and Hamiltons of the future will create the mathematics they will need just as they have in the past. These men, though honored as mathematicians, were physicists".* And further we read in [6]: *"Because the establishment in the mathematical community favors pure mathematics, the best applied work is now being done by scientists in electrical engineering, computer science, biology, physics, chemistry, and astronomy."* The "Mathematics of Harmony" presented in this book is one kind of such "new mathematics."

2.2. *The Mathematics of Harmony*

Appearance of the mathematics of harmony [100] as a new interdisciplinary direction of modern science is a natural sum of more than 2.5 millennia way of mathematics development from Euclid's *Elements* to contemporary mathematics and computer science. As we mentioned above, the mathematics of harmony takes its origin from the Euclidean "problem of division in extreme and mean ratio" (Theorem II.11) named later the Golden Section. This problem was formulated by Euclid with a purpose to give a full geometric theory of Platonic Solids (The Book XIII), in particular, Dodecahedron, which was considered in the ancient science as the main geometric figure of the Universe. By using the numerical characteristics of the Dodecahedron (12, 30, 60, 360=12×30) the Egyptians had created their calendar, which is a prototype of modern calendar. During 2.5 millennia, the mathematics of harmony was enriched by new mathematical knowledge. In the 13th century the Italian mathematician Leonardo from Pisa (Fibonacci) did very big contribution into the mathematics of harmony. He had introduced into mathematics the first in history of mathematics recursive relation generating Fibonacci numbers. This recursive relation is important contribution to the development of the combinatorial analysis. In Renaissance an interest in the golden section and harmony problem intensively increases. Just in this period the first book on the “Golden Section” appears. We are talking about the book *De Divina Proportione,* written by the outstanding Italian mathematician Luca Pacioli. Leonardo da Vinci drew 60 wonderful figures for this book published in 1509. In the 17-th century the important contribution to development of the mathematics of harmony was made by two outstanding astronomers Kepler and Cassini. Kepler used *Platonic Solids* for designing the original model of the Solar System in his 1596 *Mysterium Cosmographicum.* In his 1619 *Harmonices Mundi* he attempted to explain the proportions of the natural world - particularly the astronomical and astrological aspects - in terms of music (*Music of Spheres*). In the 17th century the French astronomer Cassini had deduced the wonderful identity for Fibonacci numbers called *Cassini Formula.* In the 19th century in works of the French mathematicians Lucas and Binet a number of the important results in the field of the golden section and Fibnacci numbers had been obtained. In particular, Lucas had introduced into mathematics *Lucas Numbers* and Binet had deduced the famous *Binet Formulas.*

In the second half of the 20th century the interest in regular polyhedrons, in the golden section and Fibonacci numbers sharply increases in theoretical natural sciences. Here a fundamental role of the Platonic and Archimedean solids (quasi-crystals, fullerenes, etc.) is realized more and more. A development of new mathematical theory - *Fibonacci Number Theory* [13, 16, 28, 38] - was a reflection of this interest in mathematics. Actually, modern Fibonacci numbers theory is a continuation of the 19th century Lucas and Binet's researches. The Russian mathematician Nikolay Vorobyov, the author of the well-known book *Fibonacci Numbers* [13], and the American mathematician Verner Hoggatt [16] made the greatest contribution to the development of this important direction in the 20th century. In 1963 Verner Hoggatt had created the Fibonacci Association and began to issue *The Fibonacci Quarterly,* the first in the science history scientific journal on Fibonacci numbers.

However, it is necessary to note, that the title of *Fibonacci Numbers Theory* narrows a subject of this scientific direction and aims mathematicians for study of properties of one important, but not the only numerical sequence, which underlies the natural phenomena. Thus, this title more misleads than really discloses a sense and a subject of this scientific direction - the creation of "harmonious models” of systems and processes of the world surrounding us.

Therefore, by the end of the 20th century the idea had ripened to come back to the sources of this scientific direction, that is, to the *Pythagorean Doctrine about Numerical Harmony of the Universe* and to the Euclidean *Elements* in order to give a new title of the *Mathematics of Harmony* for this mathematical direction. This idea had been presented in author’s lecture *The Golden Section and Modern Harmony Mathematics* [100] at the 7-th International Conference *Fibonacci Numbers and Their Applications* (Austria, Graz, July of 1996). During the next years, the author had concentrated all his efforts on the development of this direction and had published a number of the mathematical articles in the leading international journals [101-119]. These publications together with author’s preceding works [19-21, 24, 51, 87-99] had been laid at the base of the present book.

3. An Influence of the Harmony Mathematics on the Development of Modern Science and Education

3.1. *Mathematics and Mathematical Natural Sciences*

3.1.1. *New Recursive Relations, New Algebraic Equations and New Mathematical Constants.* In the recent decades many scientists independently one from another made generalizations of the Fibonacci numbers and the "golden mean." We are talking first of all about the generalized Fibonacci *p*-numbers given by (4.18) and (4.19) [20], which appear at the study of the diagonal sums of Pascal triangle [20]. A study of the Fibonacci *p*-numbers and the generalized "golden" algebraic equations (4.42) [20] resulted in an infinite number of new, unknown until now recursive numerical sequences, in a new class of mathematical constants - the golden *p*-proportions [20], which express mathematical properties of Pascal triangle, in a generalization of Binet formulas, in the generalized Lucas *p*-numbers [111] - a new class of recursive numerical sequences, and in the continuous functions for the Fibonacci and Lucas *p*-numbers [112].

The other generalization of Fibonacci numbers was introduced recently by Vera W. Spinadel [42], Midchat Gazale [45], Jay Kappraff [50] and other scientists. We are talking about the generalized Fibonacci *m*-numbers given by (4.240) and (4.241) [42, 45, 50, 118]. A study of the Fibonacci *m*-numbers and new generalized "golden" algebraic equation (4.247) had resulted in an infinite number of new, unknown until now recursive numerical sequences, in a new class of mathematical constants - the golden *m*-proportions, in a generalization of Binet formulas - Gazale formulas [45, 118], in the generalized Lucas *m*-numbers [118] - a new class of recursive numerical sequences, which include the classical Lucas numbers and the Pell-Lucas numbers as partial cases.

In 2007 Gokcen Kocer, Naim Tuglu and Alexey Stakhov had suggested the extension of the generalized Fibonacci *p*-numbers [154]. The recursive formula (4.295) at the seeds (4.296) defines a more general class of the recursive numerical sequences than the Fibonacci *p*-numbers or the Fibonacci *m*-numbers. Note that for the case $m=1$, the Fibonacci (p,m)-numbers $F_{p,m}(n)$ coincide with the Fibonacci *p*-numbers, and for the case $p=1$ the Fibonacci (p,m)-numbers coincide with the Fibonacci *m*-numbers. For the case of $p=1$ and $m=1$, the Fibonacci (p,m)-numbers coincide with the classical Fibonacci numbers. A study of the Fibonacci (p,m)-numbers and corresponding to them characteristic algebraic equation (4.303) resulted in an infinite number of new, unknown until now recursive numerical sequences, which include the Fibonacci *p*-numbers and the Fibonacci *m*-numbers as partial cases, in a new class of mathematical constants - the golden (p,m)-proportions, in a generalization of Binet formulas, in the generalized Lucas (p,m)-numbers [154] - a new class of recursive numerical sequences, which include the Lucas *p*-numbers and the Lucas *m*-numbers as partial cases.

Note that the generalized Fibonacci and Lucas numbers and the generalized golden proportions are of fundamental interest for contemporary mathematics and mathematical natural sciences because they are new mathematical models, which can be discovered in nature.

3.1.2. *Hyperbolic Fibonacci and Lucas Functions.* A discovery of the deep mathematical connection between Fibonacci and Lucas numbers and hyperbolic functions is one of the important mathematical achievements of the contemporary "Fibonacci numbers theory." For the first time, the English mathematician Vaida paid attention on such connection [28]. Independently one to another, the Ukrainian researchers Oleg Bodnar [37], Alexey Stakhov and Ivan Tkachenko [98] had introduced a new class of hyperbolic functions based on the "golden mean." In further this idea had been developed in the works by Alexey Stakhov and Boris Rozin [106, 116, 119].

As is shown in Chapter 5, in contrast to the classical hyperbolic functions, the hyperbolic Fibonacci and Lucas functions (5.53)-(5.56) have "discrete" analogs in the form of the classical Fibonacci and Lucas numbers. It is proved in Chapter 5 that every "continuous" identity for the hyperbolic Fibonacci and Lucas functions has its own "discrete" analog in the form of the corresponding identity for the classical Fibonacci and Lucas numbers. This means that the "discrete" theory of Fibonacci numbers [13, 16, 28] is particular, "discrete" case of the more general, "continuous" theory of hyperbolic Fibonacci and Lucas functions. Thus, the introduction of the hyperbolic Fibonacci and Lucas functions is raising "Fibonacci numbers theory" on a much higher scientific level.

The hyperbolic Fibonacci and Lucas functions possess unique mathematical properties (see Chapter 5). One part of these properties given by Table 5.3 is similar to the properties of the classical Fibonacci and Lucas numbers. Such properties of the hyperbolic Fibonacci and Lucas functions are called **"recursive properties."** Another part of these properties given by Theorems 5.7-5.16 is similar to the properties of the classical hyperbolic functions and is called **"hyperbolic properties."**

By discussing a "physical" sense of the hyperbolic Fibonacci and Lucas functions, we address to the modern discovery made by the Ukrainian researcher Oleg Bodnar [37]. By using the hyperbolic Fibonacci functions, he had developed an original geometric theory of phyllotaxis and explained why Fibonacci spirals arise on the surface of the phyllotaxis objects (pine cones, cacti, pine apple, heads of sunflower and so on) in process of their growths. "Bodnar's geometry" [37] confirms that the hyperbolic Fibonacci and Lucas functions are "natural" functions of the nature, which show their value in the botanic phenomenon of phyllotaxis. This fact allows us to assert that **the hyperbolic Fibonacci and Lucas functions can be attributed to the class of fundamental mathematical discoveries of contemporary science because they reflect natural phenomena, in particular, phyllotaxis phenomenon.**

3.1.3. *Gazale Formulas and a General Theory of Hyperbolic Functions.* Recently, the Egyptian mathematician Midchat Gazale [45], by studying the recursive relation (4.240) for the Fibonacci *m*-numbers, had deduced the remarkable formula (4.281) named in [118] **Gazale formula for the Fibonacci *m*-numbers.** Also the **Gazale formula for the Lucas *m*-numbers** (4.292) is deduced in [118]. Note that the Gazale formulas (4.281) and (4.292) are a wide generalization of the Binet formulas for the classical Fibonacci and Lucas numbers.

The most important outcome of the researches [118] is that "Gazale formulas" (4.281) and (4.292) resulted in a general theory of hyperbolic functions [118]. A general class of the hyperbolic functions - **the hyperbolic Fibonacci and Lucas *m*-functions** - is defined by the formulas (5.103)-(5.106). The formulas (5.103)-(5.106) give an infinite number of hyperbolic models of Nature because every real number $m>0$ originates its own class of the hyperbolic functions (5.103)-(5.106). As is proved in Chapter 5, these functions have, on the one hand, the "hyperbolic" properties similar to the properties of the classical hyperbolic functions, on the other hand, the "recursive" properties similar to the properties of the Fibonacci and Lucas *m*-numbers. In particular, the classical hyperbolic functions are partial case of the hyperbolic Lucas *m*-functions. For the case $m_e = e/2 - 2/e \approx 0.623382...$, the classical hyperbolic functions are connected with the hyperbolic Lucas *m*-functions by the correlations (5.131).

Note that for the case $m=1$, the hyperbolic Fibonacci and Lucas *m*-functions (5.103)-(5.106) coincide with the symmetric hyperbolic Fibonacci and Lucas functions (5.53) and (5.56), which were introduced by Alexey Stakhov and Boris Rozin in [106, 116]. Above we noted that the functions (5.53) and (5.56) can be attributed to the fundamental mathematical results of modern science because they reflect botanic phenomenon of phyllotaxis. It is obviously that this general conclusion can be true for the general class of the hyperbolic Fibonacci and Lucas *m*-functions (5.103)-(5.106) introduced in [118]. These functions give a general theory of hyperbolic functions what is of fundamental importance for contemporary mathematics and theoretical physics. We can suppose that the hyperbolic Fibonacci and Lucas *m*-functions [118], which correspond to the different values of *m*, can model different physical phenomena. This idea opens new ways in the development of mathematical natural sciences.

A general theory of hyperbolic functions given by (5.103)-(5.106) [118] can lead to the following scientific theories of fundamental character: **(1) Lobachevski's "golden" geometry; (2) Minkovski's "golden" geometry as original interpretation of Einstein's special relativity theory.** In Lobachevski's "golden" geometry and Minkovski's "golden" geometry, the processes of real world are modeled, in general case, by the hyperbolic Fibonacci and Lucas *m*-functions

(5.103)-(5.106). Lobachevski's geometry, Minkovski's geometry and Bodnar's geometry [37] are partial cases of this general hyperbolic geometry. We can suppose that such approach is of great importance for contemporary mathematics and theoretical physics and could become a source of new scientific discoveries.

3.1.4. *Fibonacci and "Golden" Matrices.* Since Hoggatt's book [16], a matrix approach is used widely in the "Fibonacci numbers theory." We are talking about the *Fibonacci Q-matrix* (6.2). In the form (6.4) the *Q*-matrix shows its connection with the classical Fibonacci numbers, which are elements of the matrix (6.4). The *Q*-matrix (6.2) is a generating matrix for the classical Fibonacci numbers. Alexey Stakhov [103] had developed a theory of the generating matrices for the Fibonacci *p*-numbers. They are called the *Fibonacci* Q_p*-matrices* and given by (6.28). In the form (6.30) the Q_p-matrices show their connection with the Fibonacci *p*-numbers. Also Alexey Stakhov had introduced in [118] a concept of the Fibonacci G_m-matrix (6.70), which is a generating matrix for the Fibonacci *m*-numbers. In the form (6.72) the G_m-matrix shows its connection with the Fibonacci *m*-numbers. At last, in this book a theory of the *Fibonacci* $Q_{p,m}$*-matrices* is developed. They are generating matrices for the Fibonacci (*p,m*)-numbers.

The Fibonacci *Q*-, Q_p-, G_m-, and $Q_{p,m}$-matrices and their powers possess an unique mathematical property. Their determinants are equal to +1 or -1. This unique property unites together all Fibonacci matrices and their powers into a special class of square matrices, which are of fundamental interest for matrix theory [158].

However, the "golden" matrices (6.128) and (6.129) that are originated from the *Q*-matrices (6.120) and (6.121) are also of great interest for matrix theory. A peculiarity of the "golden" matrices (6.128) and (6.129) is the fact that the continuous functions - the hyperbolic Fibonacci functions (6.122) and (6.123) - are elements of these matrices. This means that the "golden" matrices (6.128) and (6.129) are the functions of continuous variable *x*. These matrices possess a unique mathematical property (6.137) and (6.138), that is, their determinants are equal to +1 or -1.

The "golden" G_m-matrices (6.148) and (6.149) are a wide generalization of the "golden" matrices (6.128) and (6.129). They are originated from the G_m-matrices (6.146) and (6.147) and are also of great interest for matrix theory. Similarly to the "golden" matrices (6.128) and (6.129), the determinants of the "golden" G_m-matrices (6.148) and (6.149) are equal to +1 or -1.

3.1.5. *Algorithmic Measurement Theory.* As is well known, a discovery of incommensurable segments caused the first crisis in the mathematics foundations. In order to overcome this crisis, the Great mathematician Eudoxus had developed *Mathematical Theory of Magnitudes*, which underlie *Mathematical Measurement Theory* [3, 4]**.** Cantor's axiom based on the concept of *actual infinity* was introduced into this theory in the 19th century. As was shown in [20], Cantor's axiom with its "actual infinity" is the major reason why the classical measurement theory is internally contradictory theory. In author's book [20], a constructive approach to mathematical measurement theory was developed. The essence of the approach consists in the following. The measurement theory has to be constructed on the constructive idea of potential infinity. According to this idea, the measurement is considered as a procedure, which is performed during finite, but potentially unlimited number of steps. Such approach puts forward in the foreground a **problem of the synthesis of the optimal measurement algorithms.** Note that this problem never was solved in mathematics and for the first time was put forward in [20]. A proof of the existence of an infinite number of new, unknown until now optimal measurement algorithms, in particular, **Fibonacci's measurement algorithms**, is the major result of the **constructive (algorithmic) measurement theory**. At present, the algorithmic measurement theory can be used as a source of new, unknown until now positional number systems what is of great importance for computer science.

3.1.6. *A New Geometric Definition of Number.* The first definition of a number was made in the Greek mathematics. We are talking about the *Euclidean definition* of natural number given by (9.55). In spite of utmost simplicity of the *Euclidean Definition* (9.55), it had played a great role in mathematics, in particular, in number theory. This definition underlies many important mathematical concepts, for example, the concept of the *Prime* and *Composed* numbers, and also the concept of *Divisibility* that is one of the major concepts of number theory. Within many centuries, mathematicians developed and made more exact the concept of a number. In the 17th century, in period of the creation of new science, in particular, new mathematics, a number of methods of the "continuous" processes study was developed and the concept of a real number again goes out on the foreground. Most clearly, a new definition of this concept is given by Isaac Newton, one of the founders of mathematical analysis, in his *Arithmetica Universalis* (1707):

"We understand a number not as the set of units, however, as the abstract ratio of one magnitude to another magnitude of the same kind taken for the unit."

This formulation gives us a general definition of real numbers, rational and irrational. For example, the binary definition of real number given by (9.3) is the example of *Newton's Definition*, when we chose the number 2 for the unit and represent a real number as the sum of the powers of the number 2.

Similarly to the binary definition (9.3) we can use Bergman's number system (9.1), which represents a real number as the sum of the powers of the golden mean, and the sum (9.5) – the codes of the golden *p*-proportion - as new definitions of real number. Such approach is of great importance for the theory of numbers and can lead to a new theory of real numbers based on the golden mean and the golden *p*-proportion [105]. A new property of natural numbers – the ***Z*-property** – and new positional methods of natural number representation – the ***F*- and *L*-codes –** confirm a fruitfulness of such approach to the theory of numbers [105]. These results are of great importance for computer science and could become a source of new computer projects.

3.2. *Computer Science*

3.2.1. *Fibonacci Codes, Fibonacci Arithmetic and Fibonacci Computers.* A concept of *Fibonacci Computers* suggested by the author in the speech *Algorithmic Measurement Theory and Foundations of Computer Arithmetic* presented on the joint meeting of the Computer and Cybernetics Societies of Austria (Vienna, March 1976) and described in the book [20] is one of the important ideas of modern computer science. The essence of the concept consists of the following. Modern computers are based on the binary system (8.10), which represents all numbers as the sums of the binary numbers with binary coefficients, 0 and 1. However, the binary system (8.10) is non-redundant what does not allow to detect errors, which could appear in computer in the process of its exploitation. In order to eliminate this shortcoming, the author suggested [20] to use the *Fibonacci p-codes* (8.9), which represent all numbers as the sums of the Fibonacci *p*-numbers with binary coefficients, 0 and 1. In contrast to the binary number system (8.10), the Fibonacci *p*-codes (8.9) are redundant positional methods of number representation. This redundancy can be used for checking different transformations of numerical information in computer, including arithmetical operations. The original computer project - *Fibonacci Noise-Tolerant Computer* - is developed in Chapter 8. In contrast to fault-tolerant computer, the noise-tolerant computer allows to detect random malfunctions in computer.

Fibonacci computer project had been developed by the author in the former Soviet Union since 1976 right up to disintegration of the Soviet Union in 1991. 65 foreign patents of USA, Japan, England, France, Germany, Canada and other countries are official juridical documents [120-131], which confirm the Soviet priority in the Fibonacci computers.

3.2.2. *Ternary Mirror-symmetrical Arithmetic.* Computers can be constructed by using different number systems. The ternary computer "Setun" designed in Moscow University in 1958 was the first computer based not on binary system but on ternary system. The ternary mirror-symmetrical number system [104] is original synthesis of the classical ternary system [180] and Bergman's system [144]. It represents integers as the sum of the golden mean squares with ternary coefficients {-1, 0, 1}. Each ternary representation consists of two parts that are disposed symmetrically with respect to the 0th digit. However, one part is mirror-symmetrical to another part. At the increasing of number its ternary mirror-symmetrical representation is expanding symmetrically to the left and to the right with respect to 0-th digit. This unique mathematical property originates very simple method for checking numbers in computers. It is proved that the mirror-symmetric property is invariant with respect to arithmetical operations, that is, the results of all arithmetical operations have mirror-symmetrical form. This means that the mirror-symmetrical arithmetic can be used for designing the self-checked and fault-tolerant processors and computers.

The article *Brousentsov's Ternary Principle, Bergman's Number System and Ternary Mirror-Symmetrical Arithmetic* [104] published in "The Computer Journal" (England) got a high approval of the two outstanding computer specialists - **Donald Knut,** Professor-Emeritus of Stanford University and the author of the famous book *The Art of Computer Programming* [181], and **Nikolay Brousentsov,** Professor of Moscow University, a principal designer of the fist ternary computer "Setun". And this fact gives hope that the ternary mirror-symmetrical arithmetic [104] can become a source of new computer projects in the nearest time.

3.2.3. *A New Theory of Error-correcting Codes Based on the Fibonacci Matrices.* The error-correcting codes [177, 182] are used widely in modern computer and communication systems for protection of information from noises. The formula (11.2) shows that the coefficient of potential correcting ability diminishes potentially to 0 as the number n of data bits increases. For example, the Hamming (15,11)-code allows to detect $2^{11}\times(2^{15} - 2^{11})$ = 62 914 560 erroneous transitions; however, the code can correct only $2^{15} - 2^{11}$ = 30720 erroneous transitions, that is, it can correct only 30720/62 914 560 = 0, 0004882 (0, 04882%) erroneous transitions. If we take n=20, then according to (11.2) the potential correcting ability of the error-correcting (k,n)-code diminishes to 0,00009%. Thus, a potential correcting ability of the classical error-correcting codes [177, 182] is very low. This conclusion is of fundamental character and concerns all classical error-correcting codes! One more fundamental shortcoming of all known error-correcting codes is the fact that the very small information elements, bits and their combinations, are the objects of detection and correction.

A new theory of error-correcting codes based on the Fibonacci matrices [113] has a number of the important advantages in comparison to the existing theory of algebraic error-correcting codes [177, 182]: (1) the Fibonacci coding/decoding method comes to matrix multiplication, that is, to the well-known algebraic operation that is fulfilled very well in modern computers; (2) the main practical peculiarity of the Fibonacci encoding/decoding method is the fact that large information units, in particular, matrix elements, are objects of detection and correction of errors; (3) the simplest Fibonacci coding/decoding method (p=1) can guarantee a restoration of all "erroneous" (2×2)-code matrices having "single," "double" and "triple" errors; (4) the potential correcting ability of the method for the simplest case p=1 is between 26.67% and 93.33% what exceeds the potential correcting ability of all well-known algebraic error-correcting codes in 1 000 000 and more times. This means that new coding theory based on matrix approach is of great practical importance for modern computer science and opens new ways for the designing of super-reliable communication systems.

3.2.4. *The "Golden" Cryptography.* All existing cryptographic methods and algorithms [183]-[186] were created for the "ideal conditions" when we assume that coder, communication channel, and decoder operate "ideally," that is, the coder carries out the "ideal" transformation of plaintext into ciphertext, the communication channel transmits "ideally" ciphertext from the sender to the receiver and the decoder carries out the "ideal" transformation of ciphertext into plaintext. It is clear that the smallest breach of the "ideal" transformation or transmission is a catastrophe for the cryptosystem. All existing cryptosystems based on both symmetric and public-key cryptography have essential shortcoming because they do not have in their principles and algorithms the inner "checking relations" that allows checking the informational processes in the cryptosystems. The well-known public key cryptography is especially vulnerable to "noise attack."

The "golden" cryptography [118] based on the use of the special "golden" matrices (11.119)-(11.122) possesses a unique mathematical properties (11.123) and (11.124) that connect the determinants of the initial matrix (plaintext) and the code matrix (ciphertext). Thanks to these properties we can check all informational processes in cryptosystem, including encryption, decryption and transmission of the ciphertext via the channel. Such approach can result in the designing of simple for technical realization and reliable cryptosystems. **Thus, the "golden" cryptography [118] opens a new stage of the cryptography development - designing super-reliable "hybrid" cryptosystems.**

3.3. *The Golden Section, the Harmony Idea and Education*

3.3.1. *The Postulates of the "Harmony Science."* The idea of harmony, which can be traced back to the Pythagorean doctrine about numerical harmony of the Universe, is one of the most ancient scientific paradigm arisen at the same period, as well as a science. This idea belongs to the category of the "eternal" scientific problems, interest to which never died away in a science, but especially increased during the periods of the highest flowering of human culture. **There is all the grounds to believe, that the last quarter of the 20-th century and the beginning of the 21-st century became the periods of the Renaissance of this most ancient scientific paradigm in modern science.** The modern science, in which processes of differentiation prevail, requires the interdisciplinary, consolidated and synthesizing scientific discipline, which would unite all branches of a science, art and technology. *The Doctrine of Harmony* can become such interdisciplinary scientific direction**.** The following scientific postulates underlie this scientific direction:

1. The harmony reigns all over the world, it is the creative and regulating beginning of all nature and space.
2. All nature and art is the expediently and harmoniously arranged whole. Both in the nature and art, the separate things and phenomena exist as a part of the whole, as the moment in the general system of beauty and harmony.
3. The "Mathematical Harmony" is objective and general property of the Universe as a whole and its any part separately. All structures of the nature aspire to "harmonious," that is, "optimal" (from some point of view) state.

It seems that a dramatic history of the golden section, which went on during millennia, can end with great triumph for the golden section in the beginning of the 21st century, the "Century of Harmony." *Penrose's Tiles, the Resonant Theory of the Solar system* (Molchanov, Butusov), *Quasi-crystals* (Shechtman), *Fullerenes* (Curl, Kroto andSmalley, the Nobel Prize 1996) became only harbingers of this triumph. *The Mathematics of Harmony* (Stakhov), *Hyperbolic Fibonacci and Lucas Functions* (Stakhov, Tkachenko, Rozin), *Bodnar's Geometry*, *Soroko's Law of Structural Harmony of Systems*, *El Nashie's E-infinity, Fibonacci and "Golden" Matrices* (Stakhov) and, at last, *Prtoukhov's "Golden" Genomatrices* - this is far not the full list of the modern discoveries based on the golden section. These discoveries give us the ground to assume, that the golden section is some "metaphysical" knowledge, *an Universal Code of the Nature*, which can become a basis for the further development of science, in particular, mathematics, theoretical physics, genetics, computer science. This new point of view on the golden section based on the contemporary scientific discoveries can influence on the mathematical and general education.

3.3.2. *The Mathematics of Harmony, the Golden Section and Mathematical Education.* By discussing a content of mathematical education in high school, we should note that the Euclidean *Elements* are the main source of this education. What mathematical knowledge's are widely known to any educated person? Almost for certain, the "Pythagorean Theorem" - the most well-known theorem of geometry - place oneself at the head of this list. As is known, in the 17th century Johannes Kepler had named two the most important "treasures" of geometry - "Pythagorean Theorem" and "a problem of division in extreme and mean ratio" (the golden section). If we follow historical logic, we could expect that the golden section should occupy the same important place in mathematical education, as well as the "Pythagorean Theorem." Unfortunately, it is not so. During historical development the modern "materialistic" pedagogics had thrown out the golden section and the "harmony idea" on the dump of the doubtful scientific concepts together with astrology and others esoteric sciences. As a result, a majority of people well know "Pythagorean Theorem" but have rather dim idea about the golden section, one of the "treasures" of geometry. **In the 21st century, the "Century of Harmony," the ideas of harmony and the golden section should be widely introduced into the system of mathematical and general education of high schools, colleges and universities. And the scientific discipline of "The Mathematics of Harmony" should play a main role in the future reform of mathematical and general education on the base of the ideas of harmony and the golden section.**