Chapter 07.03

Simpson’s 1/3 Rule of Integration

After reading this chapter, you should be able to
1. derive the formula for Simpson's 1/3 rule of integration,
2. use Simpson's 1/3 rule it to solve integrals,
3. develop the formula for multiple-segment Simpson’s 1/3 rule of integration,
4. use multiple-segment Simpson’s 1/3 rule of integration to solve integrals, and
5. derive the true error formula for multiple-segment Simpson’s 1/3 rule.

What is integration?
Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson’s 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.

Here, we will discuss the Simpson’s 1/3 rule of approximating integrals of the form

\[ I = \int_{a}^{b} f(x) \, dx \]

where
\[ f(x) \] is called the integrand,
\[ a = \text{lower limit of integration} \]
\[ b = \text{upper limit of integration} \]

Simpson’s 1/3 Rule
The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson’s 1/3 rule is an
extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

![Figure 1 Integration of a function](image)

**Method 1:**

Hence

\[
I = \int_a^b f(x) dx \approx \int_a^b f_2(x) dx
\]

where \( f_2(x) \) is a second order polynomial given by

\[
f_2(x) = a_0 + a_1x + a_2x^2.
\]

Choose

\[
(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \text{ and } (b, f(b))
\]

as the three points of the function to evaluate \( a_0, a_1 \) and \( a_2 \).

\[
f(a) = f_2(a) = a_0 + a_1a + a_2a^2
\]

\[
f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2
\]

\[
f(b) = f_2(b) = a_0 + a_1b + a_2b^2
\]

Solving the above three equations for unknowns, \( a_0, a_1 \) and \( a_2 \) give

\[
a_0 = \frac{a^2f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2f(a)}{a^2 - 2ab + b^2}
\]

\[
a_1 = \frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}
\]
\[
\alpha_2 = \frac{2 \left( f(a) - 2f\left(\frac{a+b}{2}\right) + f(b) \right)}{a^2 - 2ab + b^2}
\]

Then
\[
I \approx \int_{a}^{b} f_2(x) \, dx
\]
\[
= \int_{a}^{b} \left( a_0 + a_1 x + a_2 x^2 \right) \, dx
\]
\[
= \left[ a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \right]_{a}^{b}
\]
\[
= a_0 (b-a) + a_1 \frac{b^2-a^2}{2} + a_2 \frac{b^3-a^3}{3}
\]
Substituting values of \(a_0\), \(a_1\) and \(a_2\) give
\[
\int_{a}^{b} f_2(x) \, dx = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]
Since for Simpson 1/3 rule, the interval \([a, b]\) is broken into 2 segments, the segment width \(h = \frac{b-a}{2}\)

Hence the Simpson’s 1/3 rule is given by
\[
\int_{a}^{b} f(x) \, dx \approx \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]
Since the above form has 1/3 in its formula, it is called Simpson’s 1/3 rule.

**Method 2:**
Simpson’s 1/3 rule can also be derived by approximating \(f(x)\) by a second order polynomial using Newton’s divided difference polynomial as
\[
f_2(x) = b_0 + b_1 (x-a) + b_2 (x-a) \left( x - \frac{a+b}{2} \right)
\]
where
\[
b_0 = f(a)
\]
\[
b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{a+b - a}
\]
\[
b_2 = \frac{f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right) - f(a)}{a+b - a}
\]
Integrating Newton’s divided difference polynomial gives us

\[
\int_a^b f(x)dx \approx \int_a^b f_2(x)dx
\]

\[
= \int_a^b \left[ b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right) \right] dx
\]

\[
= \left[ b_0x + b_1\left(\frac{x^2}{2} - ax\right) + b_2\left(\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}\right) \right]\bigg|_a^b
\]

\[
= b_0(b-a) + b_1\left(\frac{b^2-a^2}{2} - a(b-a)\right)
\]

\[
+ b_2\left(\frac{b^3-a^3}{3} - \frac{(3a+b)(b^2-a^2)}{4} + \frac{a(a+b)(b-a)}{2}\right)
\]

Substituting values of \(b_0\), \(b_1\), and \(b_2\) into this equation yields the same result as before

\[
\int_a^b f(x)dx \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]

\[
= \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]

**Method 3:**
One could even use the Lagrange polynomial to derive Simpson’s formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

\[
f_2(x) = \frac{(x - \frac{a+b}{2})(x-b)}{(a - \frac{a+b}{2})(a-b)} f(a) + \frac{(x-a)(x-b)}{(a+b - a)(a+b - b)} f\left(\frac{a+b}{2}\right) + \frac{(x-a)}{(b-a)(b-a+b)} f(b)
\]

Integrating this function gets
Simpson’s 1/3 Rule of Integration

$$\int_{a}^{b} f(x)dx = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Method 4:
Simpson’s 1/3 rule can also be derived by the method of coefficients. Assume

$$\int_{a}^{b} f(x)dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

Let the right-hand side be an exact expression for the integrals $\int_{a}^{b} dx$, $\int_{a}^{b} xdx$, and $\int_{a}^{b} x^2dx$. This implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now

$$\int_{a}^{b} dx = b - a = c_1 + c_2 + c_3$$
\[
\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1a + c_2 \frac{a+b}{2} + c_3b
\]
\[
\int_a^b x^2dx = \frac{b^3 - a^3}{3} = c_1a^2 + c_2 \left(\frac{a+b}{2}\right)^2 + c_3b^2
\]

Solving the above three equations for \(c_0\), \(c_1\) and \(c_2\) give

\[
c_1 = \frac{b-a}{6}
\]
\[
c_2 = \frac{2(b-a)}{3}
\]
\[
c_3 = \frac{b-a}{6}
\]

This gives

\[
\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)
\]

\[
= \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]

\[
= \frac{h}{3} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]

The integral from the first method

\[
\int_a^b f(x)dx \approx \int_a^b (a_0 + a_1x + a_2x^2)dx
\]

can be viewed as the area under the second order polynomial, while the equation from Method 4

\[
\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)
\]

can be viewed as the sum of the areas of three rectangles.

**Example 1**

The distance covered by a rocket in meters from \(t = 8\) s to \(t = 30\) s is given by

\[
x = \int_8^{30} \left[ 20000 \ln\left( \frac{140000}{140000 - 2100t} \right) - 9.8t \right] dt
\]

a) Use Simpson’s 1/3 rule to find the approximate value of \(x\).

b) Find the true error, \(E_t\).

c) Find the absolute relative true error, \(|\varepsilon_t|\).
Simpson’s 1/3 Rule of Integration

Solution

a) \[ x \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \]

\[ a = 8 \]
\[ b = 30 \]
\[ \frac{a+b}{2} = 19 \]

\[ f(t) = 2000\ln\frac{140000}{140000 - 2100t} - 9.8t \]

\[ f(8) = 2000\ln\frac{140000}{140000 - 2100(8)} - 9.8(8) = 177.27\text{ m/s} \]

\[ f(30) = 2000\ln\frac{140000}{140000 - 2100(30)} - 9.8(30) = 901.67\text{ m/s} \]

\[ f(19) = 2000\ln\frac{140000}{140000 - 2100(19)} - 9.8(19) = 484.75\text{ m/s} \]

\[ x \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \]

\[ = \left(\frac{30-8}{6}\right)[f(8) + 4f(19) + f(30)] \]

\[ = \frac{22}{6}[177.27 + 4 \times 484.75 + 901.67] \]

= 11065.72 m

b) The exact value of the above integral is

\[ x = \int_{8}^{30} 2000\ln\frac{140000}{140000 - 2100t} - 9.8t \, dt \]

= 11061.34 m

So the true error is

\[ E_t = True\ Value - Approximate\ Value \]

= 11061.34 - 11065.72

= -4.38 m

c) Absolute Relative true error,

\[ |\varepsilon| = \left| \frac{True\ Error}{True\ Value} \right| \times 100 \]

\[ = \left| \frac{-4.38}{11061.34} \right| \times 100 \]
Multiple-segment Simpson’s 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval \([a, b]\) into \(n\) segments and apply Simpson’s 1/3 rule repeatedly over every two segments. Note that \(n\) needs to be even. Divide interval \([a, b]\) into \(n\) equal segments, so that the segment width is given by

\[
h = \frac{b - a}{n}.
\]

Now

\[
\int_{a}^{b} f(x)\,dx = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} f(x)\,dx
\]

where

\[
x_0 = a, \quad x_n = b
\]

\[
\int_{a}^{b} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \ldots + \int_{x_{n-4}}^{x_{n-2}} f(x)\,dx + \int_{x_{n-2}}^{x_n} f(x)\,dx
\]

Apply Simpson’s 1/3rd Rule over each interval,

\[
\int_{a}^{b} f(x)\,dx \approx (x_2 - x_0) \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + (x_4 - x_2) \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \ldots
\]

\[
+ (x_{n-2} - x_{n-4}) \left[ \frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + (x_n - x_{n-2}) \left[ \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right]
\]

Since

\[
x_i - x_{i-2} = 2h, \quad i = 2, 4, \ldots, n
\]

then

\[
\int_{a}^{b} f(x)\,dx \approx 2h \left[ \frac{f(x_0) + 4f(x_1) + f(x_2)}{6} \right] + 2h \left[ \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} \right] + \ldots
\]

\[
+ 2h \left[ \frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6} \right] + 2h \left[ \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6} \right]
\]

\[
= \frac{h}{3} \left[ f(x_0) + 4\{f(x_1) + f(x_3) + \ldots + f(x_{n-1})\} + 2\{f(x_2) + f(x_4) + \ldots + f(x_{n-2})\} + f(x_n) \right]
\]
Example 2

Use 4-segment Simpson’s 1/3 rule to approximate the distance covered by a rocket in meters from \( t = 8 \) s to \( t = 30 \) s as given by

\[
x = \int_{8}^{30} \left[ 2000 \ln \left( \frac{140000}{140000 - 2100t} \right) - 9.8t \right] dt
\]

a) Use four segment Simpson’s 1/3rd Rule to find the probability.

b) Find the true error, \( E_t \), for part (a).

c) Find the absolute relative true error, \( \frac{|E_t|}{x} \), for part (a).

Solution:

a) Using \( n \) segment Simpson’s 1/3 rule,

\[
x \approx \frac{b - a}{3n} \left[ f(t_0) + 4 \sum_{i=1}^{n-1} f(t_i) + 2 \sum_{i=2}^{n-2} f(t_i) + f(t_n) \right]
\]

\[
n = 4 \\
a = 8 \\
b = 30 \\
h = \frac{b - a}{n} \\
= \frac{30 - 8}{4} \\
= 5.5
\]

\[
f(t) = 2000 \ln \left( \frac{140000}{140000 - 2100t} \right) - 9.8t
\]

So

\[
f(t_0) = f(8) \\
f(8) = 2000 \ln \left( \frac{140000}{140000 - 2100(8)} \right) - 9.8(8) = 177.27 \text{ m/s}
\]

\[
f(t_1) = f(8 + 5.5) = f(13.5)
\]
f(13.5) = 2000 \ln \left[ \frac{140000}{140000 - 2100(13.5)} \right] - 9.8(13.5) = 320.25 m/s

f(t_2) = f(13.5 + 5.5) = f(19)

f(19) = 2000 \ln \left[ \frac{140000}{140000 - 2100(19)} \right] - 9.8(19) = 484.75 m/s

f(t_3) = f(19 + 5.5) = f(24.5)

f(24.5) = 2000 \ln \left[ \frac{140000}{140000 - 2100(24.5)} \right] - 9.8(24.5) = 676.05 m/s

f(t_4) = f(t_n) = f(30)

f(30) = 2000 \ln \left[ \frac{140000}{140000 - 2100(30)} \right] - 9.8(30) = 901.67 m/s

x = \frac{b - a}{3n} \left[ f(t_0) + 4 \sum_{i=1, i \text{ odd}}^{n-1} f(t_i) + 2 \sum_{i=2, i \text{ even}}^{n-2} f(t_i) + f(t_n) \right]

= \frac{30 - 8}{3(4)} \left[ f(8) + 4 \sum_{i=3, i \text{ odd}}^{3} f(t_i) + 2 \sum_{i=2, i \text{ even}}^{2} f(t_i) + f(30) \right]

= \frac{22}{12} \left[ f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30) \right]

= \frac{11}{6} \left[ f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30) \right]

= \frac{11}{6} \left[ 177.27 + 4(320.25) + 4(676.05) + 2(484.75) + 901.67 \right]

= 11061.64 m

b) The exact value of the above integral is

x = \int_{8}^{30} \left( 2000 \ln \left[ \frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt

= 11061.34 m

So the true error is

E_t = True Value - Approximate Value

E_t = 11061.34 - 11061.64

= -0.30 m
c) Absolute Relative true error,

\[ |\varepsilon_i| = \frac{|\text{True Error}|}{|\text{True Value}|} \times 100 \]

\[ = \left| \frac{-0.3}{11061.34} \right| \times 100 \]

\[ = 0.0027\% \]

Table 1  Values of Simpson’s 1/3 rule for Example 2 with multiple-segments

| n  | Approximate Value | \( E_i \) | \(|\varepsilon_i|\) |
|----|------------------|----------|----------------|
| 2  | 11065.72         | -4.38    | 0.0396\%       |
| 4  | 11061.64         | -0.30    | 0.0027\%       |
| 6  | 11061.40         | -0.06    | 0.0005\%       |
| 8  | 11061.35         | -0.02    | 0.0002\%       |
| 10 | 11061.34         | -0.01    | 0.0001\%       |

Error in Multiple-segment Simpson’s 1/3 rule

The true error in a single application of Simpson’s 1/3rd Rule is given\(^1\) by

\[ E_i = -\frac{(b - a)^3}{2880} f^{(4)}(\zeta), \quad a < \zeta < b \]

In multiple-segment Simpson’s 1/3 rule, the error is the sum of the errors in each application of Simpson’s 1/3 rule. The error in the \( n \) segments Simpson’s 1/3rd Rule is given by

\[ E_1 = -\frac{(x_2 - x_0)^3}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2 \]

\[ = -\frac{h^5}{90} f^{(4)}(\zeta_1) \]

\[ E_2 = -\frac{(x_4 - x_2)^3}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4 \]

\[ = -\frac{h^5}{90} f^{(4)}(\zeta_2) \]

\[ \vdots \]

\[ E_i = -\frac{(x_{2i} - x_{2(i-1)})^3}{2880} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i} \]

\[ = -\frac{h^5}{90} f^{(4)}(\zeta_i) \]

\[ \vdots \]

\(^1\) The \( f^{(4)} \) in the true error expression stands for the fourth derivative of the function \( f(x) \).
\[ E_{n-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)} \left( \frac{\zeta_{n-1}}{2} \right), \quad x_{n-4} < \zeta_{n-1} < x_{n-2} \]

\[ = -\frac{h^5}{90} f^{(4)} \left( \frac{\zeta_{n-1}}{2} \right) \]

\[ E_n = -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)} \left( \frac{\zeta_n}{2} \right), \quad x_{n-2} < \zeta_n < x_n \]

Hence, the total error in the multiple-segment Simpson’s 1/3 rule is

\[ = -\frac{h^5}{90} f^{(4)} \left( \frac{\zeta_n}{2} \right) \]

\[ E_i = \sum_{i=1}^{n} E_i \]

\[ = -\frac{h^5}{90} \sum_{i=1}^{\frac{n}{2}} f^{(4)} (\zeta_i) \]

\[ = -\frac{(b - a)^5}{90n^5} \sum_{i=1}^{\frac{n}{2}} f^{(4)} (\zeta_i) \]

\[ = -\frac{(b - a)^5}{90n^4} \frac{\sum_{i=1}^{n} f^{(4)} (\zeta_i)}{n} \]

The term \( \frac{\sum_{i=1}^{n} f^{(4)} (\zeta_i)}{n} \) is an approximate average value of \( f^{(4)} (x) \), \( a < x < b \). Hence

\[ E_i = -\frac{(b - a)^5}{90n^4} f^{(4)} \]

where

\[ f^{(4)} = \frac{\sum_{i=1}^{n} f^{(4)} (\zeta_i)}{n} \]