An Approach to the solutions of time varying linear dynamic systems with multi - point boundary values on time scales

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Abstract

In this work,firstly we consider the solution of an initial value problem (IVP) of a non-homogeneous time varying linear dynamic system on a time scale $\mathbf{T}$.That is

$$x^{\Delta}(t) = A(t)x(t) + f(t), x(t_0) = x_0, t \in \mathrm{T}, x_0 \in R^N, \quad \textbf{(IVP)}$$

.Then as a main result, we obtain the solution of the following two-point boundary value (TPBVP) problem with two separated boundary values on time scales:

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + f(t) & , t \in [t_0, t_1]_T \\ Lx(t_0) = \varphi; R(t_1) = \psi \end{cases} \qquad \textbf{(TPBVP)}$$

After that it is extended to multipoint BVP. Finally we extend all these the results for non-separated BVPs
.

***Keywords**:* Time scales; Time varying; Linear system; Two - point boundary values problem, Multi-point boundary values problem.

1. Introduction

In recent years, a theory known as dynamic systems on time scales has been built which incorporates both continuous and discrete times, namely, time as an arbitrary closed sets of reals, and permit us to handle and extend both systems simultaneously.It is also avoids [1]. This theory allows one to get some insight into and better understanding of subtle differences between discrete and continuous systems.

Two-point boundary-value problems (TPBVPs) and multi-point boundary-value problems(MPBVPs) play an important role in the theory of differential and difference equations

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and in the various applications of this theory, in particular, in problems of optimal control [2,3] . Up to this time, several computational methods and applications of them in various forms have been developed for solving TPBVPs for systems [4,5,6].
In [7] and [8] existence and uniqueness of the solution of **(TPBVP)** ,when the matrix ***A*** is constant (in autonomous case), is considered for continuous and discrete cases separately.
In this article, firstly we have mentioned the needed definitions and theorems, then as main results, we obtained solutions and proved existence and uniqueness theorems for them of time varying dynamic systems with TPBVPs on arbitrary time scales and we extend theese results to MPBVPs under the boundary conditions of separated and non separated cases.

2. Preliminaries

First we give some aspects about Time Scales, then some needed lemmas and theorems are mentioned. The material in this section is drawn mainly from [1] and [12].

P.1. *Time Scales*

A ***time scale*** **T** is an arbitrary nonempty closed subset of the real numbers **R**. Thus time scale can be any of the usual integer subsets (e.g. **Z** or **N**), the entire real line **R**, or any combination of discrete points of union with continuous intervals.
The ***forward jump operator*** of T, $\sigma(t): T \to T$, is given by $\sigma(t) = \inf_{s\in T}\{s > t\}$. The ***backward jump operator*** of T, $\rho(t): T \to T$, is given by $\rho(t) = \sup_{s\in T}\{s < t\}$. The ***graininess function*** $T \to [0,\infty)$ is given by $\mu(t) = \sigma(t) - t$. Here we adopt the conventions $\inf \phi = \sup T$ (i.e. $\sigma(t) = t$ if **T** has a maximum element t), and $\sup \phi = \inf T$ (i.e. $\rho(t) = t$ if **T** has a minimum element t).For notational purposes, the intersection of a real interval [a, b] with a time scale **T** is denoted by [a, b]$\bigcap T := [a,b]_T$. A point $t \in T$ is ***right- scattered*** if $\sigma(t) > t$ and ***right- dense*** if $\sigma(t) = t$. A point $t \in T$ is ***left- scattered*** if $\rho(t) < t$ and ***left -dense*** if $\rho(t) = t$. If *t* is both left scattered and right scattered, we say that *t* is ***isolated***. If t is both left dense and right dense, we say that *t* is ***dense***. The set T^K is defined as follows: if T has a left scattered maximum *m*, then $T^K = T - \{m\}$; otherwise $T^K = T$. If $f: T \to R$ is a function, then the composition $f(\sigma(t))$ is often denoted by $f^\sigma(t)$.
For $f: T \to R$ and $t \in T^K$, define $f^\Delta(t)$ as the number (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of *t* such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \le \varepsilon\left|\sigma(t) - s\right|, \forall s \in U.$$

The function $f^\Delta : T^K \to R$ is called the **delta** derivative or the **Hilger** derivative of f on T^K. We say f is delta differentiable on T^K provided $f^\Delta(t)$ exists for all $t \in T^K$.
The following theorem establishes several important observations regarding delta derivatives.

Theorem 2.1. Suppose $f: T \to R$ and $t \in T^K$.
(i) If f is delta differentiable at *t*, then f is continuous at *t*.
(ii) If f is continuous at *t* and *t* is right- scattered, then f is delta differentiable at *t* and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If *t* is right- dense, then f is delta differentiable at *t* if and only if $\lim_{s\to t} \dfrac{f(t) - f(s)}{t - s}$ exists. In this case, $f^\Delta(t) = \lim_{s\to t} \dfrac{f(t) - f(s)}{t - s}$.

(iv) If f is delta differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.

Note that $f^{\Delta}(t)$ is precisely $f'(t)$ from the usual calculus when $T = R$. On the other hand, $f^{\Delta} = \Delta f = f(t+1) - f(t)$ (i.e. the forward difference operator) on the time scale $T = R$. These are but two very special (and rather simple) examples of time scales. Moreover, the realm of differential equations and difference equations can now viewed as but special, particular cases of more general dynamic equations on time scales, i.e. equations involving the delta derivative(s) of some unknown function.
A function $f : T \to R$ is rd-continuous if f is continuous at every right- dense point $t \in T$, and its left hand limit exists at each left- dense point. The set of rd-continuous functions $f : T \to R$ will be denoted by $C_{rd} = C_{rd}(T) = C_{rd}(T, R)$. A function $F : T \to R$ is called a (delta) antiderivative of $f : T \to R$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in T^K$. The Cauchy integral or definite integral is given by $\int_a^b f(t)\Delta t = F(b) - F(a)$, for all $a, b \in T$, where F is any (delta) antiderivative of f. Suppose that $\sup T = \infty$. Then the improper integral is defined to by $\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} F(t) \big|_a^b$ for all $a \in T$. We remark that the delta integral can be defined in terms of a Lebesgue type integral [9] or a Riemann integral [1].

Theorem 2.2 (Existence of antiderivatives).

(i) Every rd-continuous function has an antiderivative. If $t_0 \in T$, then $F(t) = \int_{t_0}^{t} f(\tau)\Delta\tau, t \in T$, is an antiderivative of f.

(ii) If $f \in C_{rd}$ and $t \in T^K$, then $\int_t^{\sigma(t)} f(\tau)\Delta\tau = f(t)\mu(t)$.

(iii) Suppose $a, b \in T$ and $f \in C_{rd}$.

(a) If $T = R$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$ (the usual Riemann integral).

(b) If $[a,b]_T$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)_T} f(t)\mu(t) & , a < b, \\ 0 & , a = b, \\ -\sum_{t\in[a,b)_T} f(t)\mu(t) & , a > b. \end{cases}$$

The last result, in above, reveals that in the continuous case , $T = R$, definite integrals are the usual Riemann integrals from calculus. When $T = Z$, definite integrals correspond to definite sums from the difference calculus; see [10].

P.2. *The Hilger's complex plane*

For $h > 0$, define the Hilger complex numbers (C_h), the Hilger real axis (R_h), the Hilger alternating axis (A_h), and the Hilger imaginary circle (I_h) by

$$C_h := \left\{ z \in C : z \neq -\frac{1}{h} \right\}, R_h = \left\{ z \in R : z > -\frac{1}{h} \right\},$$

$$A_h := \left\{ z \in R : z < -\frac{1}{h} \right\}, I_h = \left\{ z \in C : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\},$$

respectively. For $h = 0$, let $C_0 = C, R_0 = R, A_0 = \Theta,$ and $I_0 = iR$. See Fig. 1.

Let $h > 0$ and $z \in C_h$. The Hilger real part of z is defined by $\mathrm{Re}_h(z) := \frac{|zh+1|-1}{h}$, and the Hilger imaginary part of z is defined by $\mathrm{Im}_h(z) := \frac{Arg(zh+1)}{h}$, where $Arg(z)$ denotes the principal argument of z (i.e. $-\pi < Arg(z) \leq \pi$). See Fig. 1.

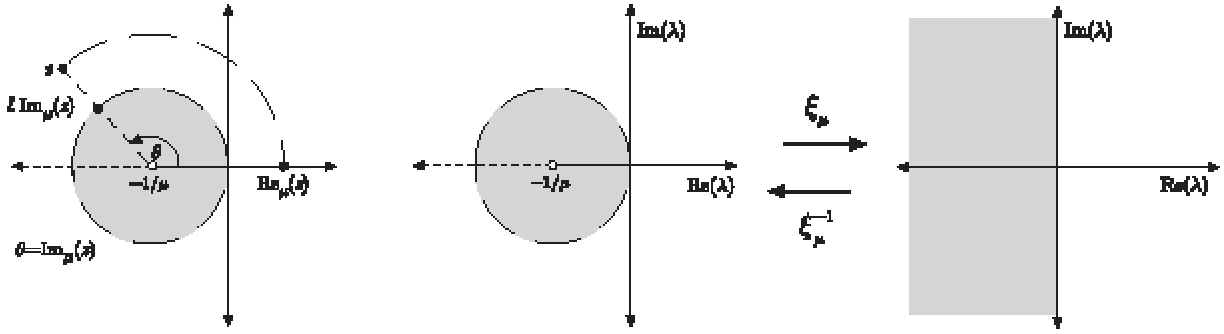

Figure 1.Left: The Hilger complex plane. Right: The cylinder**(P.1)** and inverse cylinder**(P.2)** transformations map the familiar stability region in the continuous case to the interior of the Hilger circle in the general time scale case.

For $h > 0$, define the strip $Z_h := \left\{ z \in C : -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h} \right\}$, and for $h = 0$, set $Z_0 = C$. Then we can define the cylinder transformation $\xi_h : C_h \to Z_h$ by

$$\xi_h(z) = \frac{1}{h} Log(1 + zh), z > 0, \qquad \textbf{(P.1)}$$

where Log is the principal logarithm function. When $h = 0$, we define $\xi_0(z) = z,$ for all $z \in C$. It then follows that the inverse cylinder transformation $\xi_h^{-1} : Z_h \to C_h$ is given by

$$\xi_h^{-1}(z) = \frac{e^{zh} - 1}{h}. \qquad \textbf{(P.2)}$$

See Figure 1.

P.3. Generalized exponential functions

The function $p : T \to R$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in T^K$, and this concept motivates the definition of the following sets:

$\mathfrak{R} = \{ p : T \to R : p \in C_{rd}(T) \text{ and } 1 + \mu(t)p(t) \neq 0, \ \forall t \in T^K \},$

$\mathfrak{R}^+ = \{ p \in R : 1 + \mu(t)p(t) > 0 \text{ for all } t \in T^K \}.$

The function $p:T\to R$ is *uniformly regressive* on T if there exists a positive constant δ such that $0<\delta^{-1}\le |1+\mu(t)p(t)|, t\in T^K$. A matrix is regressive if and only if all of its eigenvalues are regressive or Equivalently, if and only if $I+\mu(t)A(t)$ is invertible for all $t\in T^K$.
If $p\in\mathfrak{R}$, then we define the ***generalized time scale exponential function*** by

$$e_p(t,s)=\exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \text{ for all } s,t\in T.$$

The following theorem is a compilation of properties of $e_p(t,t_0)$ (some of which are counterintuitive) that we need in the main body of the paper.

Theorem 2.3. The function $e_p(t,t_0)$ has the following properties:

(i) If $p\in\mathfrak{R}$ then $e_p(t,r)\,e_p(r,s)=e_p(t,s)$ for all $r,s,t\in T$.
(ii) $e_p(\sigma(t),s)=(1+\mu(t)p(t))e_p(t,s)$.
(iii) If $p\in\mathfrak{R}^+$, then $e_p(t,t_0)>0$ for all $t\in T$.
(iv) If $1+\mu(t)p(t)<0$ for some $t\in T^K$, then $e_p(t,t_0)\,e_p(\sigma(t),t_0)<0$.
(v) If $T=R$ then $e_p(t,s)=e^{\int_s^t p(\tau)d\tau}$. Moreover, if p is constant, then $e_p(t,s)=e^{p(t-s)}$.
(vi) If $T=Z$, then $e_p(t,s)=\prod_{\tau=s}^{t-1}(1+p(\tau))$. Moreover, if $T=hZ$, with $h>0$ and p is constant, then $e_p(t,s)=\prod_{\tau=s}^{t-1}(1+hp(\tau))^{(t-s)/h}$.

If $p\in\mathfrak{R}$ and $f:T\to R$ is rd-continuous, then the dynamic equation

$$y^{\Delta}(t)=p(t)y(t)+f(t) \qquad \textbf{(P.3)}$$

is called regressive.

Theorem 2.4. *(Variation of constants)*. Let $t_0\in T$ and $y(t_0)=y_0\in R$. Then the regressive IVP **(P.3)** has a unique solution $y:T\to R^n$ given by

$$y(t)=e_p(t,t_0)y_0+\int_{t_0}^t e_p(t,\sigma(\tau))f(\tau)\Delta\tau.$$

We say the $n\times 1$ -vector-valued system

$$y^{\Delta}(t)=A(t)y(t)+f(t) \qquad \textbf{(P.4)}$$

is regressive provided that $A\in\mathfrak{R}$ and $f:T\to R^n$ is a rd-continuous vector-valued function.
Let $t_0\in T$ and assume that A is a $n\times n$-matrix-valued function. The unique matrix-valued solution to the IVP

$$Y^{\Delta}(t)=A(t)Y(t),\qquad Y(t_0)=I_n, \qquad \textbf{(P.5)}$$

where I_n is the $n\times n$-identity matrix, is called the transition matrix and it is denoted by $\Phi_A(t,t_0)$.
In this work, we denote the solution to **(P.5)** as $\Phi_A(t,t_0)$ when *A(t)* is time varying and denoted the solution as $e_A(t,t_0)\equiv\Phi_A(t,t_0)$ (*the exponential matrix function*, as in [1]) . The following theorem lists some properties of the transition matrix.

Theorem 2.5. Suppose $A, B \in \mathfrak{R}$ are matrix-valued function on T.

(i) Then the semigroup property $\Phi_A(t,r)\,\Phi_A(r,s) = \Phi_A(t,s)$ is satisfied for all $r,s,t \in T$.
(ii) $\Phi_A(\sigma(t),s) = (I + \mu(t)A(t))\Phi_A(t,s)$.
(iii) If $T = R$ then A is constant, then $\Phi_A(t,s) = e_A(t,s) = e^{A(t-s)}$.
(iv) If $T = hZ$, with $h > 0$ and A is constant, then $\Phi_A(t,s) = e_A(t,s) = (I + hA)^{(t-s)/h}$.

We now present a theorem that guarantees a unique solution to the regressive $n \times 1$-vector-valued dynamic IVP **(P.4)**.

Theorem 2.6. *(Variation of constants)*. Let $t_0 \in T$ and $y(t_0) = y_0 \in R^n$. Then the regressive IVP **(P.4)** has a unique solution $y : T \to R^n$ given by

$$y(t) = \Phi_A(t,t_0)y_0 + \int_{t_0}^{t} \Phi_A(t,\sigma(\tau)) f(\tau)\Delta\tau .$$

3. THE MAIN RESULTS

3.1. (Two point BVP) Let $j_1 := [t_0, t_1]_T \subseteq T$. We consider non-homogeneous time varying linear system with two - point boundary values

$$\begin{cases} x^\Delta(t) = A(t)x(t) + f(t)\ ,\ t \in \mathrm{j}_1{}^k & (D_1) \\ Lx(t_0) = \varphi \ \ ;\ \ Rx(t_1) = \psi & (D_2) \end{cases}$$

where $A(t) = (a_{ij}(t))$ a $\mathrm{N} \times \mathrm{N}$ matrix, L a $k \times \mathrm{N}$ matrix and R a $(\mathrm{N} - k) \times \mathrm{N}$ matrix are real matrices, φ, ψ are real column vectors and $f(t)$ is real column time scale function with k, (N-k) and N elements orderly.
Our first result is given by theorem 3.1. It is obvious that both the continuous case [7, Theorem 4.1] and the discrete one [8, Theorem 4.3.1] are special cases when (D_1) is autonomous.

THEOREM 3.1. Let the system (D_1) be regressive and $\Phi_A(t,t_0) = e_A(t,t_0)$ is the fundamental matrix of homogeneous part of (D1). Furthermore let W be given as $W = \Phi_A(t_1,t_0) = e_A(t_1,t_0)$.
Then we can define the matrix H as $H = \begin{pmatrix} L \\ RW \end{pmatrix}$. If the matrix H is nonsingular, then the problem $(\mathrm{D}_1\text{ -}\mathrm{D}_2)$ has the following unique solution :

$$x(t) = e_A(t,t_0)H^{-1}\begin{pmatrix} \varphi \\ \psi - R\displaystyle\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau \end{pmatrix} + \int_{t_0}^{t} e_A(t,\sigma(\tau))f(\tau)\Delta\tau \qquad \textbf{(3.1.1)}$$

It is clear that **(3.1.1)** could be written in terms of $\Phi_A(t,t_0)$ and $\Phi_A(t_1,t_0)$ as well.

3.2. (Three point BVP) Let $j_2 := [t_0, t_2]_T \subseteq T$ and $t_1 \in (t_0, t_2)_T$. Now let the system mentioned above be given with three -point separated boundary values

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + f(t) \ , \ t \in j_2{}^k & (E_1) \\ B_0 x(t_0) = \varphi_0 \ ; \ B_1 x(t_1) = \varphi_1 \ ; \ B_2 x(t_2) = \varphi_2 & (E_2) \end{cases}$$

where $A(t) = (a_{ij}(t))$ a $N \times N$ matrix, B_0 a $k_0 \times N$ matrix and B_1 a $k_1 \times N$ matrix and B_2 a $k_2 \times N$ ($k_0 + k_1 + k_2 = N$) are real matrices, $\varphi_0, \varphi_1, \varphi_2$ are real column vectors and $f(t)$ is real column time scale function with k_1, k_2, k_3 and N elements, respectively.

THEOREM 3.2. Let the system (E_1) be regressive and $\Phi_A(t,t_0) = e_A(t,t_0)$ is the fundamental matrix of homogeneous part of (E_1). Furthermore let W_1, W_2 be given as
$W_1 = \Phi_A(t_1,t_0) = e_A(t_1,t_0)$; $W_2 = \Phi_A(t_2,t_0) = e_A(t_2,t_0)$. Then we can define the matrix *H* as
$H = \begin{pmatrix} B_0 \\ B_1 W_1 \\ B_2 W_2 \end{pmatrix}$. If the matrix *H* is nonsingular, then the problem (E_1 - E_2) has the following unique solution **:**

$$x(t) = e_A(t,t_0) H^{-1} \begin{pmatrix} \varphi_0 \\ \varphi_1 - B_1 \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) f(\tau) \Delta\tau \\ \varphi_2 - B_2 \int_{t_0}^{t_2} e_A(t_2, \sigma(\tau)) f(\tau) \Delta\tau \end{pmatrix} + \int_{t_0}^{t} e_A(t, \sigma(\tau)) f(\tau) \Delta\tau \qquad \textbf{(3.2.1)}$$

We can easily extend the above theorem with help of the induction method to multi-point BVP.

3.3. (Multi-point BVP) Let $j := [t_0, t_n]_T \subseteq T$ and $t_k \in (t_0, t_n)_T$; $0 < k < n$. Now let the system mentioned above with multi - point separated boundary values

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + f(t), t \in j^k & (F_1) \\ B_i x(t_i) = \varphi_i \ ; \ 0 \le i \le n \ ; \quad \forall t_i \in [t_0, t_n] \ , \ t_i < t_{i+1} & (F_2) \end{cases}$$

where $A(t) = (a_{ij}(t))$ a $N \times N$ matrix, B_i a $k_i \times N$ matrix ($\sum_{0 \le i \le n} k_i = N$) are real matrices, φ_i $(0 \le i \le n)$ are real column vectors and $f(t)$ is real column time scale function with k_i and N elements,orderly. The solution to this problem is given by the following theorem **:**

THEOREM 3.3. Let the system (F_1) be regressive and $\Phi_A(t,t_0) = e_A(t,t_0)$ is the fundamental matrix of homogeneouspart of (F_1). Furthermore let W_i be given as

$W_i = \Phi_A(t_i, t_0) = e_A(t_i, t_0)$, $1 \le i \le n$. Then we can define the matrix H as $H = \begin{pmatrix} B_0 \\ B_1 W_1 \\ B_2 W_2 \\ \cdots \\ B_n W_n \end{pmatrix}$. If the matrix H is nonsingular, then the problem (F_1 -F_2) has the following unique solution :

$$x(t) = e_A(t,t_0)H^{-1}\begin{pmatrix} \varphi_0 \\ \varphi_1 - B_1\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau \\ \varphi_2 - B_2\int_{t_0}^{t_2} e_A(t_2,\sigma(\tau))f(\tau)\Delta\tau \\ \cdots \\ \varphi_n - B_n\int_{t_0}^{t_n} e_A(t_n,\sigma(\tau))f(\tau)\Delta\tau \end{pmatrix} + \int_{t_0}^{t} e_A(t,\sigma(\tau))f(\tau)\Delta\tau \qquad \textbf{(3.2.2)}$$

3.4. (Two point Nonseparated BVP) Let $j_1 := [t_0, t_1]_T \subseteq T$. We consider non-homogeneous time varying linear system with nonseparated two- point boundary values

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + f(t) \ , \ t \in j_1^{\ k} & (ND_1) \\ Lx(t_0) + Rx(t_1) = \alpha & (ND_2) \end{cases}$$

where $A(t) = (a_{ij}(t))$, L and R $N \times N$ real matrices α is real column vector and $f(t)$ is real column time scale function with N elements.

THEOREM 3.4. Let the system (ND_1) be regressive and $\Phi_A(t,t_0) = e_A(t,t_0)$ is the fundamental matrix of homogeneous part of (ND_1). Furthermore let W be given as $W = \Phi_A(t_1,t_0) = e_A(t_1,t_0)$. Then we can define the matrix H as $H = (L + RW)$. If the matrix H is nonsingular, then the problem (ND_1 -ND_2) has the following unique solution :

$$x(t) = e_A(t,t_0)H^{-1}(\alpha - R\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau) + \int_{t_0}^{t} e_A(t,\sigma(\tau))f(\tau)\Delta\tau \qquad \textbf{(3.3.1)}$$

3.5. (Three point Nonseparated BVP) Let $j_2 := [t_0, t_2]_T \subseteq T$ and $t_1 \in (t_0, t_2)_T$. Now let the system mentioned above with nonseparated three- point boundary values

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + f(t) \ , \ t \in j_2^{\ k} & (NE_1) \\ B_0x(t_0) + B_1x(t_1) + B_2x(t_2) = \alpha & (NE_2) \end{cases}$$

where $A(t)=(a_{ij}(t))$ a $N\times N$ matrix, B_0, B_1 and B_2 are $N\times N$ real matrices, α is real column vector and $f(t)$ is real column time scale function with N elements.

THEOREM 3.5. Let the system (NE_1) be regressive and $\Phi_A(t,t_0)=e_A(t,t_0)$ is the fundamental matrix of homogeneous part of (NE1). Furthermore let W_1, W_2 be given as
$W_1=\Phi_A(t_1,t_0)=e_A(t_1,t_0)$; $W_2=\Phi_A(t_2,t_0)=e_A(t_2,t_0)$. Then we can define the matrix H as $H=(B_0+B_1W_1+B_2W_2)$. If the matrix H is nonsingular, then the problem (NE_1 -NE_2) has the following unique solution **:**

$$x(t)=e_A(t,t_0)H^{-1}(\alpha-B_1\int_{t_0}^{t_1}e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau-B_2\int_{t_0}^{t_2}e_A(t_2,\sigma(\tau))f(\tau)\Delta\tau)+\int_{t_0}^{t}e_A(t,\sigma(\tau))f(\tau)\Delta\tau \quad \mathbf{(3.4.1)}$$

We can easily extend the above theorem, with the help of the induction method to multipoint BVP.

3.6. (Nonseparated Multi-point BVP) Let $j:=[t_0,t_n]_T\subseteq T$ and $t_k\in(t_0,t_n)_T$; $0<k<n$. Now let the system, mentioned above, be given with nonseparated multi-point boundary values

$$\begin{cases} x^{\Delta}(t)=A(t)x(t)+f(t) \quad , \quad t\in j^k & (NF_1)\\ \sum_{0\le i\le n} B_i x(t_i)=\alpha \qquad \forall t_i\in[t_0,t_n]\,,\, t_i<t_{i+1} & (NF_2)\end{cases}$$

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where $A(t)=(a_{ij}(t))$ a $N\times N$ matrix, B_i are $N\times N$ real matrices, α is real column vector and $f(t)$ is real column time scale function with N elements.

THEOREM 3.6. Let the system (NF_1) be regressive and $\Phi_A(t,t_0)=e_A(t,t_0)$ is the fundamental matrix of homogeneous part of (NF_1). Furthermore let W be given as
$W_i=\Phi_A(t_i,t_0)=e_A(t_i,t_0)$,$1\le i\le n$. Then we can define the matrix H as
$H=(B_0+B_1W_1+B_2W_2+\cdots+B_nW_n)$. If the matrix H is nonsingular, then the problem (NF_1 -NF_2) has the following unique solution :

$$x(t)=e_A(t,t_0)H^{-1}\Bigg(\alpha-B_1\int_{t_0}^{t_1}e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau\varphi_2-B_2\int_{t_0}^{t_2}e_A(t_2,\sigma(\tau))f(\tau)\Delta\tau- \\ \ldots-B_n\int_{t_0}^{t_n}e_A(t_n,\sigma(\tau))f(\tau)\Delta\tau\Bigg)+\int_{t_0}^{t}e_A(t,\sigma(\tau))f(\tau)\Delta\tau \quad \mathbf{(3.4.2)}$$

4. THE PROOFS OF THE THEOREMS

Proof of Theorem 3.1.

(**Existence**): From (2.4) in Part 2 we know that, the geneneral solution of (D_1) can be written as

$$x(t) = e_A(t,t_0)C + \int_{t_0}^{t} e_A(t,\sigma(\tau))f(\tau)\Delta\tau \qquad \textbf{(4.1)}$$

Here C is the value of $x(t_0)$ which is unknown, so if we apply **(4.1)** in equations of (D_2), we will have

$$Lx(t_0) = \varphi \quad \Rightarrow \quad Le_A(t_0,t_0)C + \int_{t_0}^{t_0} e_A(t_0,\sigma(\tau))f(\tau)\Delta\tau = \varphi \;\Rightarrow\; LC = \varphi \qquad \textbf{(4.2)}$$

$$Rx(t_1) = \psi \quad \Rightarrow \quad R(e_A(t_1,t_0)C + \int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau) = \psi \quad \Rightarrow$$

$$\operatorname{Re}_A(t_1,t_0)C = \psi - R\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau \quad . \qquad \textbf{(4.3)}$$

Now, from **(4.2)** and **(4.3)** one obtains

$$\begin{pmatrix} L \\ \operatorname{Re}_A(t_1,t_0) \end{pmatrix} C = \begin{pmatrix} L \\ R\Phi_A(t_1,t_0) \end{pmatrix} C = \begin{pmatrix} \varphi \\ \psi - R\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau \end{pmatrix} \Rightarrow$$

$$C = H^{-1}\begin{pmatrix} \varphi \\ \psi - R\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau \end{pmatrix}$$

By replacing the value of C in **(4.1)** the proof of existence will be finished.

(Uniqueness): Suppose $x_1(t)$ and $x_2(t)$ be two different solutions for (D_1 -D_2), so if we put $z(t) = x_2(t) - x_1(t)$,

then with the help of (D_1 -D_2), we can write

$$\begin{cases} z^{\Delta}(t) = A(t)Z(t), \; t \in j^k \\ Lz(t_0) = 0 \;\;,\;\; Rz(t_1) = 0 \end{cases}$$

Therefore from **(3.1.1)**, concluding $z(t) \equiv 0$, gives $x_2(t) \equiv x_1(t)$ which is a contradiction with being

difference of $x_1(t)$ and $x_2(t)$.So the proof of the uniqueness is completed.▲

Proof of Theorem 3.2.

(**Existence**): Starting the general solution with **(4.1)** so that C is the value of $x(t_0)$ which is unknown, so if we apply **(4.1)** in equations of (E_2), we can have

$$B_0 x(t_0) = \varphi_0 \quad \Rightarrow \quad B_0(e_A(t_0,t_0)C + \int_{t_0}^{t_0} e_A(t_0,\sigma(\tau))f(\tau)\Delta\tau) = \varphi_0 \;\Rightarrow\; B_0 C = \varphi_0 \quad \textbf{(4.4)}$$

$$B_1 x(t_1) = \varphi_1 \quad \Rightarrow \quad B_1(e_A(t_1,t_0)C + \int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau) = \varphi_1 \quad \Rightarrow$$

$$B_1 e_A(t_1,t_0)C = \varphi_1 - B_1\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau \quad . \qquad \textbf{(4.5)}$$

$$B_2 x(t_2) = \varphi_2 \quad \Rightarrow \quad B_2(e_A(t_2,t_0)C + \int_{t_0}^{t_2} e_A(t_2,\sigma(\tau))f(\tau)\Delta\tau) = \varphi_2 \quad \Rightarrow$$

$$B_2 e_A(t_2,t_0)C = \varphi_2 - B_2\int_{t_0}^{t_2} e_A(t_2,\sigma(\tau))f(\tau)\Delta\tau \quad . \qquad \textbf{(4.6)}$$

Now, from **(4.4) (4.5)** and **(4.6)** one obtains the following result :

$$\begin{pmatrix} B_0 \\ B_1 e_A(t_1,t_0) \\ B_2 e_A(t_2,t_0) \end{pmatrix} C = \begin{pmatrix} B_0 \\ B_1 \Phi_A(t_1,t_0) \\ B_2 \Phi_A(t_2,t_0) \end{pmatrix} C = \begin{pmatrix} \varphi_0 \\ \varphi_1 - B_1\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau \\ \varphi_2 - B_2\int_{t_0}^{t_2} e_A(t_2,\sigma(\tau))f(\tau)\Delta\tau \end{pmatrix} \quad \Rightarrow$$

$$C = H^{-1}\begin{pmatrix} \varphi_0 \\ \varphi_1 - B_1\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau \\ \varphi_2 - B_2\int_{t_0}^{t_2} e_A(t_2,\sigma(\tau))f(\tau)\Delta\tau \end{pmatrix}$$

By replacing the value of C in **(4.1)** the proof of existence will be finished.
(Uniqueness): It would be obtained in the same way as theprevious theorem.▲

Proof of Theorem 3.3.

(Existence): Starting the general solution with **(4.1)** so that C is the value of $x(t_0)$ which is unknown, so if we apply **(4.1)** in equations of (F_2), we can have

$$B_i x(t_i) = \varphi_i \quad \Rightarrow \quad B_i(e_A(t_i,t_0)C + \int_{t_0}^{t_i} e_A(t_i,\sigma(\tau))f(\tau)\Delta\tau) = \varphi_i \quad \Rightarrow$$

$$B_i e_A(t_i,t_0)C = \varphi_i - B_i\int_{t_0}^{t_i} e_A(t_i,\sigma(\tau))f(\tau)\Delta\tau \quad . \quad 0 \le i \le n \qquad \textbf{(4.7)}$$

Now, from **(4.7)** we can easily find the following:

$$\begin{pmatrix} B_0 \\ B_1 e_A(t_1,t_0) \\ B_2 e_A(t_2,t_0) \\ \cdots \\ B_n e_A(t_n,t_0) \end{pmatrix} C = \begin{pmatrix} B_0 \\ B_1 \Phi_A(t_1,t_0) \\ B_2 \Phi_A(t_2,t_0) \\ \cdots \\ B_n \Phi_A(t_n,t_0) \end{pmatrix} C = \begin{pmatrix} \varphi_0 \\ \varphi_1 - B_1 \int_{t_0}^{t_1} e_A(t_1,\sigma(\tau)) f(\tau)\Delta\tau \\ \varphi_2 - B_2 \int_{t_0}^{t_2} e_A(t_2,\sigma(\tau)) f(\tau)\Delta\tau \\ \cdots \\ \varphi_n - B_n \int_{t_0}^{t_n} e_A(t_n,\sigma(\tau)) f(\tau)\Delta\tau \end{pmatrix} \Rightarrow$$

$$C = H^{-1} \begin{pmatrix} \varphi_0 \\ \varphi_1 - B_1 \int_{t_0}^{t_1} e_A(t_1,\sigma(\tau)) f(\tau)\Delta\tau \\ \varphi_2 - B_2 \int_{t_0}^{t_2} e_A(t_2,\sigma(\tau)) f(\tau)\Delta\tau \\ \cdots \\ \varphi_n - B_n \int_{t_0}^{t_n} e_A(t_n,\sigma(\tau)) f(\tau)\Delta\tau \end{pmatrix}$$

By replacing the value of C in **(4.1)** the proof of the existence will be finished.

(Uniqueness): It would be obtained in the same way as the previous theorem.▲

Proof of Theorem 3.4.

From (2.4) in Part 2, general solution of (ND_1) could be written as

$$x(t) = e_A(t,t_0)C + \int_{t_0}^{t} e_A(t,\sigma(\tau)) f(\tau)\Delta\tau \qquad \textbf{(4.8)}$$

C is the value of $x(t_0)$ which is unknown, so if we apply **(4.8)** in equations of (ND_2), we have

$$Lx(t_0) + Rx(t_1) = \alpha \quad \Rightarrow \quad \begin{cases} L(e_A(t_0,t_0)C + \int_{t_0}^{t_0} e_A(t_0,\sigma(\tau)) f(\tau)\Delta\tau) + \\ \quad R(e_A(t_1,t_0)C + \int_{t_0}^{t_1} e_A(t_1,\sigma(\tau)) f(\tau)\Delta\tau) = \alpha \end{cases}$$

$$\Rightarrow\ (L + \mathrm{Re}_A(t_1,t_0))C = \alpha - R\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau)) f(\tau)\Delta\tau$$

$$\Rightarrow C = H^{-1}\left(\alpha - R\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau)) f(\tau)\Delta\tau \right)$$

By replacing the value of C in **(4.8)** the proof of the existence will be finished.

Uniqueness of the solution is obvious.▲

Proof of Theorem 3.5.

Similarly by using **(4.8)** as general solution of (NE_1) which C is the value of $x(t_0)$ which is unknown, so if we apply **(4.8)** in equations of (NE_2), we have

$$B_0x(t_0)+B_1x(t_1)+B_2x(t_2)=\alpha \quad \Rightarrow \begin{cases} B_0(e_A(t_0,t_0)C+\int_{t_0}^{t_0} e_A(t_0,\sigma(\tau))f(\tau)\Delta\tau)+ \\ B_1(e_A(t_1,t_0)C+\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau)+ \\ B_2(e_A(t_2,t_0)C+\int_{t_0}^{t_2} e_A(t_2,\sigma(\tau))f(\tau)\Delta\tau)=\alpha \end{cases}$$

$$\Rightarrow \begin{cases} (B_0+B_1e_A(t_1,t_0)+B_2e_A(t_2,t_0))C= \\ \alpha-B_1\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau-B_2\int_{t_0}^{t_2} e_A(t_2,\sigma(\tau))f(\tau)\Delta\tau \end{cases}$$

$$\Rightarrow C=H^{-1}\left(\alpha-B_1\int_{t_0}^{t_1} e_A(t_1,\sigma(\tau))f(\tau)\Delta\tau-B_2\int_{t_0}^{t_2} e_A(t_2,\sigma(\tau))f(\tau)\Delta\tau\right)$$

By replacing the value of C in **(4.8)** the proof of the existence willbe finished.

Uniqueness of the solution is obvious.▲

Proof of Theorem 3.6.

Similarly by using **(4.8)** as general solution of (NF) which C is the value of $x(t_0)$ which is unknown, so if we apply **(4.8)** in equations of (NF2), we have

$$\sum_{0\le i\le n} B_ix(t_i)=\alpha \quad \Rightarrow \sum_{0\le i\le n} B_i(e_A(t_i,t_0)C+\int_{t_0}^{t_i} e_A(t_i,\sigma(\tau))f(\tau)\Delta\tau)=\alpha$$

$$\Rightarrow (B_0+\sum_{1\le i\le n} B_i(e_A(t_i,t_0))C=\alpha-\sum_{1\le i\le n} B_i\int_{t_0}^{t_i} e_A(t_i,\sigma(\tau))f(\tau)\Delta\tau$$

$$\Rightarrow C=H^{-1}\left(\alpha-\sum_{1\le i\le n} B_i\int_{t_0}^{t_i} e_A(t_i,\sigma(\tau))f(\tau)\Delta\tau\right)$$

By replacing the value of C in **(4.8)** the proof of the existence will be finished.

Uniqueness of the solution is obvious.▲

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