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RESIDUAL ERROR ESTIMATION FOR THE LINEAR DIFFERENCE EQUATIONS SYSTEM HAVING CONSTANT COEFFICIENTS UNDER TWO-POINT BOUNDARY VALUES

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ABSTRACT

Nowadays solving mathematical problems with computer have been taken important affords parallel to the developments of computer technologies. Since computers are using only rational subsets of the real numbers usually have some calculating errors. In this study taking attention to these we deal with the following two-point boundary value problem (TPBVP) with seperated boundary conditions specially with the residual error estimation :

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi; Rx(n_1) = \psi; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\} \end{cases} .(*)$$

Where A $N \times N$ matrix, L $k \times N$ matrix and R $(N\text{-}k) \times N$ matrix are real matrices, φ and ψ are real column vectors of N and $N\text{-}k$ orderly. It is known that the residue of this problem can be given as $f(n) = y(n+1) - Ay(n)$ where $Ly(n_0) = \widetilde{\varphi}$, $Ry(n_1) = \widetilde{\psi}$. Therefore we obtain the following problem:

$$\begin{cases} y(n+1) = Ay(n) + f(n) \\ Ly(n_0) = \widetilde{\varphi}; Ry(n_1) = \widetilde{\psi}; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\} \end{cases}$$

Here y(n) is the computed solution by computers of the given problem.

Key words. Difference equation, two-point boundary value problem, residual error.

SABİT KATSAYILI LİNEER İKİ-NOKTA SINIR DEĞERLİ FARK DENKLEM SİSTEMİ İÇİN KALINTI HATA TAHMİNİ

ÖZET

Günümüzde bilgisayar teknolojisindeki gelişmelere paralel olarak matematiksel problemlerin bilgisayarla çözülmesi önem kazanmıştır. Bilgisayarlar sadece reel sayıların bir alt kümesi olan rasyonel sayılarla çalıştığından bilgisayarla yapılan hesaplamalarda bazı hesap hataları meydana gelir. Bu çalışmada aşağıdaki ayrılabilir sınır şartlı iki-nokta sınır değer problemi için rezidü hata tahminine dikkat çekiyoruz:

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi; Rx(n_1) = \psi; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\} \end{cases}$$

Burada A, L ve R matrisleri sırasıyla $N \times N$, $k \times N$ ve $(N\text{-}k) \times N$ tipinde reel matrisler, φ ve ψ sırasıyla N ve N-k bileşenli reel kolon vektörleridir. Bilindiği gibi bu problemin rezidu vektörü $f(n) = y(n+1) - Ay(n)$ şeklinde verilebilir. Burada $Ly(n_0) = \widetilde{\varphi}$, $Ry(n_1) = \widetilde{\psi}$ dir. Böylece aşağıdaki problemi elde ederiz:

$$\begin{cases} y(n+1) = Ay(n) + f(n) \\ Ly(n_0) = \widetilde{\varphi}; Ry(n_1) = \widetilde{\psi}; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\} \end{cases}$$

Burada y(n) çözümü verilen problemin bilgisayarda hesaplanan çözümüdür.

Anahtar Kelimeler. Fark Denklemi, iki-nokta sınır değer problemi, rezidü hatası

1. INTRODUCTION

The theory of difference equations is a lot richer than the corresponding theory of differential equations. Many authors have been studied on difference equations and some problems related with them. Such as stability theory (1), existence and uniqueness theorem (2, 7), transmission of information (3), signal processing, oscillation (9), asymptotic behavior, control and dynamic systems (10), control systems (11), etc. (see also 6). Two point boundary value problems play the important role in the theory of differential and difference equations and in various applications of this theory, in particular, in problems of optimum control. (12,13).

Conditioning of problems is very important concept in numerical analysis. One way to measure the magnification factor is by means of the quantity $\|A\|\|A^{-1}\|$ called the condition number of A. The condition number determines the loss in precision due to roundoff errors in Gaussian elimination and can be used to estimate the accuracy of results obtained from matrix inversion and linear equation solution. It arises naturally in perturbation theories that compare the perturbed solution $(A+E)^{-1}b$ with the true solution $A^{-1}b$. Another important aspect of conditioning is that, in general, residuals are reliable indicators of accuracy only if the problem is well- conditioned. The residual vector for a linear system $Ax = b$ is computed always as $r = Ax - b$. But residuals are unreliable indicators of relative solution accuracy for ill-conditioned problems. Many authors have been studied on residual vector and residual error estimation and some problems related with them (14,15,16,17,18,19).

Our aim in this article as we have mentioned at the beginning is to exam the residual error estimation related with (*). We have tried to solve the problem by getting help from solutions of the TPBVP for homogeneous and nonhomogeneous systems.

2. TWO-POINT BOUNDARY VALUE PROBLEMS FOR THE SYSTEM OF LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

Now we consider the initial value problem for the system of the nonhomogeneous difference equations

$$x(n+1)=Ax(n)+f(n), x(n_0)=x_0. \tag{2.1}$$

It is known that the solution of the problem can be given as

$$x(n) = \Psi(A, n-n_0)x(n_0) + \sum_{k=n_0}^{n-1} \Psi(A, n-k-1) f(k).$$

(3, 8). Where, Ψ *(A, n)* is the fundamental matrix of the homogeneous system of (2.1). And on the right side of the second term is a particular solution of the nonhomogeneous system.

We give the following two definitions for the sake of use in the sequel:

Definition 2.1. Let A $N\times N$ matrix, L $k\times N$ matrix and R $(N\text{-}k)\times N$ matrix be real matrices. φ, ψ and $f(n)$ are real column vectors with N, $(N\text{-}k)$ and N elements orderly. Then the following problem is called as a two-point boundary value problem for the system of the linear difference equations with constant coefficients (2):

$$\begin{cases} x(n+1) = Ax(n) + f(n) \\ Lx(n_0) = \varphi; Rx(n_1) = \psi; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\} \end{cases} \tag{2.2}$$

Definition 2.2. For an N×N matrix A the number $\mu(A) = \|A\|\,\|A^{-1}\|$ is called as the condition number of A (4,5).

2.1. Existing and Uniqueness Theorems For The TPBVP

Theorem 2.1. We consider the following homogeneous two-point boundary value problem;

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi; Rx(n_1) = \psi; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\} \end{cases} \tag{2.3}$$

Then the following two cases can be met:

Case 1. If A is a nonsingular matrix and $\psi(A,n)$ is the fundamental matrix of $x(n+1)=Ax(n)$, $n= 0,\pm1, \pm2, \pm3,\dots$. Furthermore let W be given as $W = \psi(A,n_1-n_0)\psi^{-1}(A,0)$.

Case 2. If A is a singular matrix then there exist an orthogonal matrix U such that

$$U^*AU=\begin{pmatrix} A_1 & B \\ 0 & A_0 \end{pmatrix},$$

is satisfied. Where A_0 is $N_0\times N_0$ real matrix having all eigenvalues as 0 and $\det A_1 \neq 0$. $\psi(A_1,n)$ is the fundamental matrix of $x(n+1) = A_1 x(n)$. Let W be given as

$$W= \psi(A, n_1-n_0)(\psi^{-1}(A_1,0), 0)\, U^*.$$

For these two cases we can define the matrix H as $H=\begin{pmatrix} L \\ RW \end{pmatrix}$. If the matrix H is a nonsingular matrix, then the problem has a unique solution (2).

Theorem 2.2. We consider the following nonhomogeneous TPBVP

$$\begin{cases} x(n+1) = Ax(n) + f(n) \\ Lx(n_0) = \varphi; Rx(n_1) = \psi; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\} \end{cases}$$

If the matrix H is a nonsingular matrix then there exist a unique solution of the problem and the solution given as

$$x(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi - R\sum_{k=n_0}^{n_1-1} A^{n_1-k-1} f(k) \end{pmatrix} + \sum_{k=n_0}^{n-1} A^{n-k-1} f(k)$$

(2).

3. RESIDUAL PROBLEM OF TPBVP

Our goal in this article to show that how small residue is the accuracy of the computed solution of TPBVP high. Let y(n) be computed solution of the following TPBVP obtained by using a computer:

$x(n+1) = Ax(n)$

$L\,x(n_0) = \varphi, \quad R\,x(n_1) = \psi$

Then the residual vector of y(n) will be as follows:

$$f(n) = y(n+1) - Ay(n) \qquad (3.1)$$

Ideally, we would like to have f (n) =0, but in practice f (n) ≈ 0. It is conceivable that a small residual error implies good accuracy in the computed y (n). We are interested in knowing how accurate the computed y (n) is, given f (n). When we put the computed solution y(n) in the boundary conditions we obtain the following:

$Ly(n_0) = \tilde{\varphi}, \quad Ry(n_1) = \tilde{\psi}.$

Then y(n) be the exact solution of the problem.

$y(n+1) = Ay(n) + f(n)$

$$Ly(n_0) = \tilde{\varphi}, \quad Ry(n_1) = \tilde{\psi} \qquad (3.2.)$$

Theorem 3.1. Let consider the following two problems on the set of $\{ n : n\in Z, n_o \le n \le n_1 \}$:

$x(n+1) = Ax(n)$

$Lx(n_0) = \varphi, \quad Rx(n_1) = \psi$

and

$y(n+1) = Ay(n) + f(n)$

$Ly(n_0) = \tilde{\varphi}, \quad Ry(n_1) = \tilde{\psi}.$

Where $\tilde{\varphi}, \tilde{\psi}$ are close enough to the vectors φ ve ψ in the sense of norm, orderly. In addition to the sequence {f(n)} is small enough in norm. The closeness of the solutions each other is depended on the condition number $\mu(H) = \|H\|\|H^{-1}\|$. Where

$$H = \begin{pmatrix} L \\ RA^{n_1-n_0} \end{pmatrix}$$

Then letting θ(A) be

$$\theta(A) = \max_{n_0 \le k \le n_1 - 1} \|A^{n_1-k-1}\|$$

under these conditions the following inequality holds:

$$\|x(n) - y(n)\| \le \frac{\|A^{n-n_0}\|}{\|H\|}.\mu(H).$$

$$\left(\|\varphi - \tilde{\varphi}\| + \|\psi - \tilde{\psi}\| + \|R\|.(n_1 - n_0).\theta(A). \max_{n_0 \le k \le n_1 - 1} \|f(k)\| \right)$$

$$+ (n - n_0).\theta(A). \max_{n_0 \le k \le n - 1} \|f(k)\|.$$

Proof. Let we consider the problems of

$x(n+1) = Ax(n)$

$Lx(n_0) = \varphi, \quad Rx(n_1) = \psi$

and

$y(n+1) = Ay(n) + f(n)$

$Ly(n_0) = \tilde{\varphi}, \quad Ry(n_1) = \tilde{\psi}.$

The solution of the first problem can be written as

$$x(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

and the solution of the second problem has the following form:

$$y(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} - R\sum_{k=n_0}^{n_1-1} A^{n_1-k-1} f(k) \end{pmatrix}$$

$$+ \sum_{k=n_0}^{n-1} A^{n-k-1} f(k).$$

From these the following equality can be given easily :

$$x(n) - y(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \varphi - \tilde{\varphi} \\ \psi - \tilde{\psi} + R \sum_{k=n_0}^{n_1 - 1} A^{n_1 - k - 1} f(k) \end{pmatrix}$$

$$- \sum_{k=n_0}^{n-1} A^{n-k-1} f(k).$$

By taking

$$\left\| \sum_{k=n_0}^{n-1} A^{n-k-1} f(k) \right\| \le \max_{n_0 \le k \le n-1} \| f(k) \| \sum_{k=n_0}^{n-1} \left\| A^{n-k-1} \right\| = \| F(n) \| \cdot$$

We obtain the following :

$$\|x(n) - y(n)\| \le \left\| A^{n-n_0} \right\| \left\| H^{-1} \right\| \left(\|\varphi - \tilde{\varphi}\| + \|\psi - \tilde{\psi}\| + \|R\| \|F(n)\| \right) + \|F(n)\|$$

Then,

$$\max_{n_0 \le k \le n_1 - 1} \left\| A^{n-k-1} \right\| \le \max_{n_0 \le k \le n_1 - 1} \left\| A^{n_1 - k - 1} \right\| = \theta(A)$$

can be given easily. Therefore

$$\|x(n) - y(n)\| \le \left\| A^{n-n_0} \right\| . \left\| H^{-1} \right\| .$$

$$\left(\|\varphi - \tilde{\varphi}\| + \|\psi - \tilde{\psi}\| + \|R\| . (n_1 - n_0) . \theta(\mathrm{A}) . \max_{n_0 \le k \le n_1 - 1} \| f(k) \| \right)$$

$$+ (n - n_0) \theta(A) . \max_{n_0 \le k \le n-1} \| f(k) \| .$$

Finally,

$$\|x(n) - y(n)\| \le \frac{\left\| A^{n-n_0} \right\|}{\|H\|} . \mu(H) .$$

$$\left(\|\varphi - \tilde{\varphi}\| + \|\psi - \tilde{\psi}\| + \|R\| . (n_1 - n_0) . \theta(\mathrm{A}) . \max_{n_0 \le k \le n_1 - 1} \| f(k) \| \right)$$

$$+ (n - n_0) . \theta(A) . \max_{n_0 \le k \le n-1} \| f(k) \|$$

can be obtained. This completes the proof of the theorem.

4. CONCLUSION

In this article, we have worked on the residual error estimation for the two-point boundary value problem. Here we considered the exact solution of (2.3) and y(n) be approximated solution of the same problem. After that with the help of residual vector of the given problem we examined the problem of (3.2) of which y(n) is exact solution. Thus we have obtained the difference between the exact solution and the approximated solution of the problem (2.3) in the (3.1) residual vector and input data. Finally, we have found the error estimation with respect to the changes on input data and the residual vector in detail.

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