ON A SYMMETRIZATION OF THE BOUNDARY VALUE PROBLEM

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Abstract

Two-point boundary value problem with non-separated boundary conditions for a homogeneous stationary Hamilton system of the ordinary differential equations is considered. Algorithm of factorization of a fundamental matrix of this system is offered. This algorithm is connected with construction of the stabilizing solution of the algebraic Riccati equation. Then the results are generalized for the three-point and multi-point boundary value problem. Matrices which obtained as outcome of such factorization allow to symmetrize the relation connecting the values of the phase vector in the beginning and in the end of the interval. That, in turn, allows to improve the condition number of the linear equations, which defined the unknown values of initial conditions.

The proposed algorithm defining only value of the initial condition in the beginning of the interval and LAGRANG multiplier of the corresponding initial and final conditions for the control problem. Such approach essentially decreases the dimension of Linear Algebraic Equation System defining initial and final conditions. On the base of factorization algorithm of the fundamental matrices in some cases is improved the condition number of the corresponding system of linear algebraic equations defining of the initial condition basing on the base of given here solution of corresponding differential Riccati equation. The results are illustrated on the examples.

1. Introduction.

Two-point boundary value problems play the important role in the theory of the differential equations and in various applications of this theory, in particular, in problems of optimum control [1,2,3,4], robotics [5,6] etc. It is natural , that various algorithms for solution of such problems (see, for example [1,2,3,7] and the bibliography contained there) were offered. In the elementary case the two-point problem is formed as follows [7]. It is necessary to determine x(t)- the n-dimensional vector , which satisfying to the following relations:

$$Lx = \frac{dx}{dt} - A(t)x = 0, \quad B_1 x(0) + B_2 x(\tau) = b, \qquad (1)$$

in which the matrices $A(t), B_1, B_2$, vector b and final moment τ are given, $t \in (0,\tau)$.One of algorithms of solution of this problem is the algorithm basing on construction of the fundamental matrix $X(t)$, which is defined as follows:

$$LX = 0, \quad X(0) = I. \qquad (2)$$

Hereinafter I-is an identity matrix of the appropriate size. Really, since

$$x(\tau) = X(\tau)x(0), \qquad (3)$$

that, in this case, the initial condition for the equation $Lx = 0$ can be written in an explicit form

$$x(0) = (B_1 + B_2 X(\tau))^{-1} b.$$

However, well-known, that in many cases, such approach may not be effective at numerical realization. Therefore, other algorithms often are used [1,2,7]. Really, use of the relation (3) may reduce to loss of accuracy of calculations. We shall illustrate it on the example in which the size of the vector x is equal to 2.

We accept, that $A = \begin{bmatrix} -\lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $\lambda > 0$, $B_1 = B_2 = I$. The structure of the vector b is not detailed.

Obviously, in this example

$$X(\tau) = \begin{bmatrix} e^{-\lambda} & 0 \\ 0 & e^{\lambda} \end{bmatrix}.$$

Let's consider two variants of the relations, which permitting to receive the required initial condition ($x(0)$) for the equation $Lx = 0$.

In first of them we use the relation (3):

$$X(\tau)x(0) - x(\tau) = 0,$$

$$B_1 x(0) + B_2 x(\tau) = b. \qquad (4)$$

We pay attention to the "asymmetry" of the first equation in this relation (the matrixes $X(\tau)$ and I are factors of $x(0)$ and $x(\tau)$).

In the second approach, we factorizing the matrix $X(\tau)$:

$$X(\tau) = X_+ X_-,$$

$$X_- = \begin{bmatrix} e^{-\lambda\tau} & 0 \\ 0 & 1 \end{bmatrix}, \quad X_+ = \begin{bmatrix} 1 & 0 \\ 0 & e^{\lambda\tau} \end{bmatrix}.$$

It allows to note the first equation in (4) in more "symmetric" form, namely:

$$X_{-}x(0) - X_{+}^{-1}x(\tau) = 0,$$

$$B_1x(0) + B_2x(\tau) = b. \qquad (5)$$

"Symmetry" is understood in the sense that eigenvalues modulo of matrices X_{-}, X_{+}^{-1} are less or equal to unit. Thus, for determination of $x(0)$ it is necessary to solve the set of equations (4) or (5). For comparison of an expected accuracy of determination of $x(0)$ we compare condition numbers (cond) of matrices M_1 and M_2, which are defining these equations :

$$M_1 = \begin{bmatrix} X(\tau) & -I \\ B_1 & B_2 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} X_{-} & -X_{+}^{-1} \\ B_1 & B_2 \end{bmatrix}.$$

Numerical experiments show, that for enough big values of λ, the following estimations are fair by $\tau = 1$

$$cond(M_1) \cong 1{,}62e^{\lambda}, \quad cond(M_2) \cong 2{,}62.$$

Bad conditioning of the matrix M_1 point out, that the using of this relation (3) at big λ may reduce to inadmissible errors. On the other hand, the system (4) obtained in outcome of "symmetrization" of a relation (3) may provide a high exactitude of determination of $x(0)$ and at enough big values of λ .Thus, it is exist problems, in which by the factorization of the fundamental matrix ("symmetrization" of connection between $x(0)$ and $x(1)$) it is possible to increase accuracy of outcome essentially. The approach, which permit " symmetrized" connection between $x(0)$ and $x(1)$, further will be considered. It will be assumed, that appearing in (1) matrix $A(t)$ is Hamilton.

2. A stationary Hamilton system.

In the theory of optimal control, the important place is occupied by two-point problems in which the matrix $A(t)$ is Hamilton [2,4]. Let, appearing in (1) Hamilton matrix $A(t)$ does not depend on time and has the following structure:

$$A = \begin{bmatrix} F & -GR^{-1}G' \\ -Q & -F' \end{bmatrix}. \qquad (6)$$

Hereinafter the prime means a transposition. In (6) the matrices R and Q are symmetric. As the example of such problem may serve the problem of optimization of system

$$\dot{x} = Fx + Gu \qquad (7)$$

according to the quadratic functional

$$\int_0^{\tau} (x'Qx + u'Ru)dt, \qquad (8)$$

if the values of $x(0), x(\tau)$ are prescribed [4]. In (7), (8) x, u - are a phase vector and a vector of controlling actions, accordingly, matrices F, G determine the dynamics of plant.

It is supposed, that the algebraic Riccati equation (ARE)

$$F'P + PF - PGR^{-1}G'P + Q = 0 \qquad (9)$$

has stabilizing solution [8], i.e. the solution at which matrix

$$D = F - GR^{-1}G'P \qquad (10)$$

has eigenvalues in the open left half-plane (LHP). In this case, the Lyapunov equation

$$DY + YD' + GR^{-1}G' = 0 \qquad (11)$$

has the unique solution. The matrices [9,10]

$$T = \begin{bmatrix} I - YP & Y \\ -P & I \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} I & -Y \\ P & I - YP \end{bmatrix} \qquad (12)$$

are diagonalizing the matrix A, i.e.

$$TAT^{-1} = \begin{bmatrix} D & 0 \\ 0 & -D' \end{bmatrix}$$

and, accordingly, the matrix $X(t)$, defined by (2)

$$TX(t)T^{-1} = \begin{bmatrix} e^{Dt} & 0 \\ 0 & e^{-D't} \end{bmatrix}$$

The outcome of the factorization of the matrix $X(\tau)$ can be presented as

$$X_- = T\begin{bmatrix} e^{D\tau} & 0 \\ 0 & I \end{bmatrix}T^{-1}, \quad X_+ = T\begin{bmatrix} I & 0 \\ 0 & e^{-D'\tau} \end{bmatrix}T^{-1}. \qquad (13)$$

Thus, in considered case the following analog of the relation (5) is obtained

$$X_- x(0) - X_+^{-1} x(\tau) = 0,$$
$$B_1 x(0) + B_2 x(\tau) = b, \qquad (14)$$

in which the matrices X_- and X_+ are determined by the relations (9) - (13). It is obvious, that modules of eigenvalues of matrices X_- and X_+^{-1} do not exceed unit. Let's mark, that the initial data of the example, which was considered in the introduction, in the notations accepted here look so: $F = -\lambda$, $G = 0$, $Q = 0$. Hence, the stabilizing solution of ARE (9) and the solution of the Lyapunov equation (11) will be zero, i.e. $P = 0$, $Y = 0$. Hence, $T = I$, $D = -\lambda$ and we obtain the system (5) again.

3.Multipoint problems.

We consider the similar problem in the case of the equations (1), (6) and three-point boundary value problem

$$B_0 x(0) + B_1 x(\tau_1) + B_2 x(\tau_2) = q \qquad (15)$$

in the interval $(0, \tau_2)$, where $\tau_1 \in (0, \tau_2)$. Using [11] one may get $x(\tau_1)$, $x(\tau_2)$ in the following forms

$$\overline{\overline{X}}(\tau_1) = X(\tau_1)x(0), \quad \overline{\overline{X}}(\tau_2) = X(\tau_2)x(0) \qquad (16)$$

and substituting the last in (15) for the definition of x(0) is obtained

$$x(0) = (B_0 + B_1 e^{A\tau_1} + B_2 e^{A\tau_2})^{-1} q$$

as it has been shown in point **1**, if $B_2 = 0$, $\tau_1 = \tau_2$ leads to loosing of stability, i. e. it needs modify the relations (14). For it, using the relations (9), (12) we present $X(\tau_1)$, $X(\tau_2)$ from (3), (16) in the form (13)

$$X(\tau_1) = X_{1+}X_{1-}, \qquad X(\tau_2) = X_{2+}X_{2-},$$

where

$$X_{1-} = T\begin{bmatrix} e^{D\tau_1} & 0 \\ 0 & I \end{bmatrix} T^{-1}, \quad X_{1+} = T\begin{bmatrix} I & 0 \\ 0 & e^{-D\tau_1} \end{bmatrix} \mathrm{T}^{-1}$$

$$X_{2-} = T\begin{bmatrix} e^{D\tau_2} & 0 \\ 0 & I \end{bmatrix} T^{-1}, \qquad X_{2+} = T\begin{bmatrix} I & 0 \\ 0 & e^{-D\tau_2} \end{bmatrix} T^{-1} \qquad (17)$$

Taking into account (17) in (16) and complexing the obtained relation together with (15) for the definition of x(0), $x(\tau_1), x(\tau_2)$, we get the following system of linear algebraic equations (LAE)

$$\left.\begin{array}{l} X_{1+}^{-1} x(\tau_1) - X_{1-} x(0) = 0 \\ X_{2+}^{-1} x(\tau_2) - X_{2-} x(0) = 0 \\ B_0 x(0) + B_1 x(\tau_1) + B_2 x(\tau_2) = q \end{array}\right\} \qquad (18)$$

By sufficiently large values τ_1 and $(\tau_2 - \tau_1)$ from (18), the exponential expressions are sufficiently small and modules of the eigenvalues X_{1-}, X_{1+}^{-1}, X_2^-, X_{2+}^{-1} less than unit. Therefore the conditioning of system LAE (15), (16) is much worse than (18).

We generalize the results of given case for multi-point boundary problems, i.e. in (1) instead of boundary conditions (15) is obtained [3]

$$B_0 x(o) + \sum_{i=0}^{k-1} B_i x(\tau_i) + B_k x(\tau_k) = q \quad , \qquad (19)$$

where B_i are matrices of corresponding dimensions and $\tau_i \in (\tau_0, \tau_k)$. In that case $X(\tau_i)$ is expressed through $x(0)$ by formula

$$x(\tau_i) = X(\tau_i) x(0) \qquad (20)$$

where $X(\tau_i)$ is factorized in the form

$$X(\tau_i) = X_{i+} X_{i-} ,$$

and

$$X_{i-} = T \begin{bmatrix} e^{D\tau_i} & 0 \\ 0 & I \end{bmatrix} T^{-1} \quad , \quad X_{i+} = T \begin{bmatrix} I & 0 \\ 0 & e^{-D'\tau_i} \end{bmatrix} T^{-1} \quad . \qquad (21)$$

Instead of equation (12) we have the following linear equations, defining $x(0)$, $x(\tau_i)$

$$\left.\begin{array}{l} X_{1+}^{-1} x(\tau_1) - X_{1-} x(0) = 0 \\ \vdots \\ X_{i+}^{-1} x(\tau_i) - X_{i-} x(0) = 0 \\ \vdots \\ X_{k+}^{-1} x(\tau_k) \ - X_{k-} x(0) = 0 \\ B_0 x(0) + B_1 x(\tau_1) + \ldots + B_i x(\tau_i) + \ldots + B_k x(\tau_k) = q \end{array}\right\} \qquad (22)$$

Thus, the modules of eigenvalues X_{i+}^{-1} *and* X_{i-} are not greater than unit here, too and we have the following algorithm of finding $x(\tau_i)$ and

$$x(t) = e^{At} x(\tau_i) \ t \in (\tau_i, \tau_{i+1}) \ \ i = 0,1, \ldots, k, \tau \qquad (23) .$$

Algorithm

Step1. The initial matrices A, B_i $(i = 0,1,...,k)$ are formed .

Step2. ARE (9) is solved relatively positively defined solution $P \geq o$.

Step3. Lyapunov equation L(1) is solved relatively $Y \geq 0$.

Step 4. The matrix of transformation T from (12) and X_{i+}, X_{i-} from (21) are calculated.

Step 5. The matrices B and q are formed in the form

$$
B = \begin{bmatrix} -X_{1-} & X_{1+}^{-1} & 0 \; . \; . \; . \; . & 0 \\ -X_{2-} & 0 & X_{2+}^{-1} \; . \; . \; . \; . & 0 \\ \vdots & & & \\ -X_{i-} & 0 & 0 \ldots X_{i+}^{-1} \ldots & 0 \\ \vdots & & & \\ -X_{k-} & 0 & 0 \ldots 0 \; \ldots & X_{k+}^{-1} \\ B_1 & B_2 & B_3 \ldots B_i \; \ldots & B_k \\ & & & \end{bmatrix}, \qquad \bar{q} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q \end{bmatrix}
$$

and LAE (22) is solved

$$
B \begin{bmatrix} x(0) \\ \vdots \\ x(\tau_i) \\ \vdots \\ x(\tau_k) \end{bmatrix} = \bar{q} \tag{24}
$$

relatively $x(0), x(\tau_1)...x(\tau_k)$

Step 6. The solution of multi-point boundary problem (1), (19) is defined from (23).

We mark, such algorithm may successfully realized with the help of MATLAB.

4.The method decreasing the dimension of LAE (14).

Other method [1,12] of solution of control problem with two-point non-separated boundary value conditions (7), (8), (4) is quite differ from (14), i. e. instead of linear algebraic equation, defining $x(0)$, $x(\tau)$ is suggested another linear algebraic equation

$$
\begin{bmatrix} S(0) & N(0) + B_1^{'} \\ N'(0) + B_1 & n(0) \end{bmatrix} \begin{bmatrix} x(0) \\ \nu \end{bmatrix} = \begin{bmatrix} 0 \\ q \end{bmatrix} \tag{25}
$$

defining lacking initial condition $x(0)$ and Lagranj multiplier ν, corresponding to the control problem (7), (6), (8), (14), where S, N, n are defined from following systems of differential equations

$$
\dot{S} = -F'S - SF + SMS - Q; \qquad S(\tau) = 0 \tag{26}
$$

$$
\dot{N} = -D^{'}N \qquad N(\tau) = -B_2^{'} \tag{27}
$$

$$
\dot{n} = N^{'}MN, \qquad n(\tau) = 0 \tag{28}
$$

where matrices S, N, n have corresponding dimensions.

We stop at the conditioning of matrix of system of LAE (25).

We definite the solution of differential equations of Riccati (26). For it, we use the expressions from [11] and (12). We form the fundamental matrix X of system (1), (6) in the form

$$X(\tau,t)=\begin{bmatrix} e^{D(\tau-t)}(I-YP)+Ye^{-D^{'}(\tau-t)}P & e^{D(\tau-t)}Y-Ye^{-D^{'}(\tau-t)} \\ Pe^{D(\tau-t)}(I-YP)-(I-YP)e^{-D^{'}(\tau-t)}P & Pe^{D(\tau-t)}Y+(I-YP)e^{-D^{'}(\tau-t)} \end{bmatrix}=\begin{bmatrix} X_{11}(\tau,t) & X_{12}(\tau,t) \\ X_{21}(\tau,t) & X_{22}(\tau,t) \end{bmatrix}$$

Then from [11], solution (26) S(t) and S(0) has the following form

$$S(t)=-[Pe^{D(\tau-t)}Y+(I-YP)e^{-D^{'}(\tau-t)}]^{-1}\left[Pe^{D(\tau-t)}(I-YP)-(I-YP)e^{-D^{'}(\tau-t)}P\right] \qquad (29),$$

and

$$S(0)=-X_{22}^{-1}(\tau,0)X_{21}(\tau,0)=-\left[Pe^{D\tau}Y+(I-YP)e^{-D^{'}\tau}\right]^{-1}\left[Pe^{Dt}(I-YP)-(I-YP)e^{-D^{'}t}P\right] \qquad (30)$$

We suppose τ is sufficiently large number. Then, as it has been noted above, used relation (25) may lead to inadmissible errors. However, in such case the solution $S(t)$ of differential Riccati equation (26) tents [13] to the solution of corresponding algebraic Riccati equation (9), i.e. $\frac{dS}{dt}\approx 0$ in (26) (it is clear from relations (29), (30)).

Here, the solutions (27) and (28) are not difficult, because there $SM-F^{'}=-D^{'}$ and

$$N(t)=-e^{D^{'}(\tau-t)}B_2^{'}, \quad N(0)=e^{-D^{'}\tau}B_2^{'} \qquad (31)$$

By this the matrix n(t) has the form

$$n(t)=-\int_t^{\tau}B_2e^{D(\tau-t)}Me^{D^{'}(\tau-t)}B_2^{'}dt \qquad (32)$$

and by sufficiently large τ the conditioning of system of LAE (25) is sufficiently improved.

Now, the following algorithm is obtained:

1. The matrices F,G, M,Q,R are formed.
2. Solving ARE (9) and calculating $n(0)$ using (30), the system of LAE (31) is solved. We find $x(0),\nu$. Then, we find u(t) and x(t), using [1].

We illustrate an example, given in the introduction through proposed last algorithm. Really, here (see [1])

$F = -\lambda$, $G = 0, Q = 0$, $\Phi_1 = 1, \Phi_2 = -1, q = 1$

Then $S(t) = 0, N(0) = e^{-\lambda\tau}, n(t) = 0, n(0) = 0$

and (25) turn into the form

$$L\begin{vmatrix} x(0) \\ v \end{vmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad \text{where} \quad L = \begin{vmatrix} 01 \\ 10 \end{vmatrix},$$

where Cond L = 1, i.e. this algorithm improves the conditioning of system of LAE, defining x(0).

Conclusion.

For two and multipoint non-separated boundary value problem for stationary Hamilton system the effective algorithm of solution by sufficiently large interval of time is suggested. This algorithm is based on the solution of corresponding ARE and AEL. For lacking boundary data, LAE of smaller dimensionality is given. An effectivity of the algorithm is demonstrated by the improving of the conditioning of corresponding systems LAE for the definition of initial data.

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