Algorithms of the Solutions of Linear Quadratic Optimal Control Problem with Nonseparated Boundary Conditions in Mixed (Continuous – Discrete) Typed

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Abstract.

In this article the mixed (continuous – discrete) optimal control problem with non seperated conditions is examined. In suggested algorithm the system has less number of equations than the number of equations used to solve continuous and discrete optimal control problems separately. Suggested algorithm is better than the others in the sense of computer realization. Finally we illustrated the algorithm with a concrete example.

1. Introduction

Let the interval $(0,\tau)$ is devided into subintervals of the form (τ_i,τ_{i+1}) and the motion of the object is given by [1-3]

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) + \upsilon(t) \tag{1}$$

on each (τ_i,τ_{i+1}) and at each points of τ_i is given by the discrete equations of [3-5]

$$x(\tau_i + 0) = F_{\delta_i}.x(\tau_i - 0) + \Gamma_i V(\tau_i) \tag{2}$$

Finally at the boundary points of $\{0,\tau_p + 0\}$ the motion of the object is given by [3]

$$\Phi_1 x(0) - \Phi_2 x(\tau_p + 0) = q \tag{3}$$

[3]. Here Φ_1 and Φ_2 are the constant matrices of the form of k×n (k<n). $F(t)$, $G(t)$ and $\upsilon(t)$ are the continuous matrices of the forms n×n, n×m and n×1 orderly on (τ_i, τ_{i+1}). $x(t)$ is the position vector of the object and $u(t)$ is the control vector with the forms of n×1, m×1 orderly.

It is wanted to find $u(t)$ and $V(t)$ such that the performance index

$$J = \frac{1}{2}\sum_i V'(\tau_i) C_i V(\tau_i) + \frac{1}{2}\int_0^{\tau_p - 0} [x'(t)R(t)x(t) + u'(t)C(t)u(t)]dt \tag{4}$$

is minimum under the conditions of (1), (2), (3). Here C_i are the constant matrices of the form m×m such that $C_i = C_i' > 0$. The matrices $R(t)$ and $C(t)$ are integrable on (τ_i, τ_{i+1}) with the properties of $R(t) = R'(t) \geq 0$ and $C(t) = C'(t) > 0$ which have the forms of n×n and m×m orderly. Here $R'(t)$ and $C'(t)$ are the transposes of $R(t)$ and $C(t)$ orderly.

This problem has been examined in the form of continuous and discrete cases separately in [1,3,5]. The idea of symmetrization given in [1, 2] has some difficulties. Therefore it will be better to use the sweeping algorithms [3].

In this article, firstly the problem of (1) – (4) reduced from the conditional extremum problem to the extremum problem having no restrictions. So we obtained the Euler – Lagrange Differential Equations and for the continuous case of (1), (4) and the discrete case of (2), (4) problems orderly by using the condition (3). After that we examine the properties of the same equations of which $F_{\delta_i}^{-1}$ exists or not exists. Here we give two different algorithms for the solution of this problem. In the first algorithm it requires the solution of the system of having 2n differential equations and the solution of the system of 4n linear algebraic equations. In the second algorithm it requires a suitable Riccati Matrix Differential Equations having less dimensions, linear differential equations and the system of n+k (k < n) linear algebraic equations.

The results have been shown in a simple form for the moving object on the program trajectory and under the control.

2. The Solution of The Case Being $F_{\delta_i}^{-1}$ exist.

As we have mentioned above by writing the extended performance index for the problem (1) - (4) we obtain the following system of Euler-Lagrange Equations for the continuous case [6]:

$$u(t) = -C^{-1}(t)G'(t)\lambda(t) \tag{5}$$

and

$$\begin{aligned}&\dot{x}(t) = F(t)x(t) - G(t)R^{-1}(t)G'(t)\lambda(t) + \upsilon(t)\\&\dot{\lambda}(t) = -R(t)x(t) - F'(t)\lambda(t)\end{aligned} \tag{6}$$

For the discrete case we have

$$u(\tau_i) = -C_i^{-1}\Gamma_i\lambda(\tau_i + 0) \tag{7}$$

and

$$\begin{aligned}&x(\tau_i + 0) = F_{\delta_i}x(\tau_i - 0) - M_i\lambda(\tau_i + 0)\\&\lambda(\tau_i + 0) = S_f x(\tau_i - 0) + F'_{\delta_i}\lambda(\tau_i + 0)\end{aligned} \tag{8}$$

We have the suitable boundary conditions (3) and

$$\lambda(0) = -\Phi_1\nu, \lambda(\tau_p + 0) = -\Phi'_2\nu \tag{9}$$

So far this case the optimization problem of (1) – (4) reduced to the solution of the differential system equation (6) and the discrete system (8) with the conditions (3) and (9) orderly. The optimal control can be stated in the form of (5) and (7).

Now, let us suppose that $F_{\delta_i}^{-1}$ be exist. Then we have [7]

$$\Phi' = P^{-1}\begin{bmatrix}E\\0\end{bmatrix}Q^{-1} \tag{10}$$

under the condition of $\Phi = [\Phi_1, -\Phi_2]$. After some transformations we can obtain

$$-P_3\lambda(0) + P_4\lambda(\tau_p + 0) = 0 \tag{11}$$

from (9).

Where $P = \begin{bmatrix}P_1 & P_2\\P_3 & P_4\end{bmatrix}$.

By using (3) and (11) we can obtain the following equality of

$$\begin{bmatrix}\Phi_1 & 0\\0 & P_3\end{bmatrix}\begin{bmatrix}x(0)\\\lambda(0)\end{bmatrix} + \begin{bmatrix}-\Phi_2 & 0\\0 & P_4\end{bmatrix}\begin{bmatrix}x(\tau_p + 0)\\\lambda(\tau_p + 0)\end{bmatrix} = \begin{bmatrix}q\\0\end{bmatrix} \tag{12}$$

Thus (6) with the dimension 2n and (8) with the same dimension leads us to the solution of the optimization problem of (1) – (4) under the condition (12). Such that when $F_{\delta_i}^{-1}$ be exist by using the solutions of (6) and (8) on the intervals of (τ_i, τ_{i+1}) we can write the following result:

$$\begin{bmatrix}x(\tau_p + 0)\\\lambda(\tau_p + 0)\end{bmatrix} = \Phi(0, \tau_p + 0)\begin{bmatrix}x(0)\\\lambda(0)\end{bmatrix} + V(\tau_p + 0) \tag{13}$$

Where $\Phi(0, \tau_p + 0)$ is the fundamental matrix of (6) and $V(\tau_p + 0)$ has been obtained from (8). If we apply the boundary conditions (12) in the solution of (13) we obtain

$$\left(\begin{bmatrix}\Phi_1 & 0\\0 & P_3\end{bmatrix} + \begin{bmatrix}-\Phi_2 & 0\\0 & P_4\end{bmatrix}\Phi(0, \tau_p + 0)\right)\begin{pmatrix}x(0)\\\lambda(0)\end{pmatrix} = \begin{bmatrix}q\\0\end{bmatrix} - \begin{bmatrix}-\Phi_2 & 0\\0 & P_4\end{bmatrix}V(\tau_p + 0) \tag{14}$$

for the initial condition of $\begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$. Let the coefficients matrix has inverse. So, after finding $\begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$ from the system of linear algebraic equation (14). The solution of (6) can be find as the discrete equations (8) be satisfied for the arbitrary $\tau_i \neq t_i \in [0, \tau_p + 0]$. Thus the relation between $\lambda(t_i)$ and $u(t_i)$ can be obtained as follows

$$u(t_i) = -C^{-1}(t_i)G'(t_i)\lambda(t_i).$$

Finally, adding this knowledge together we can give the following algorithm for the solution of the problem (1) – (4):

Algorithm 1.

Step 1. Construct the matrix $\Phi = [\Phi_1, -\Phi_2]$ with the help of (3). Then find P_3 and P_4 from (10).

Step 2. Write the fundamental matrix $\Phi[0, \tau_p + 0]$ by (6) and write the vector $V(\tau_p + 0)$ by (8).

Step 3. Write $\begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$ by solving the algebraic equation (4).

Step 4. Obtain $\begin{bmatrix} x(\tau_i) \\ \lambda(\tau_i) \end{bmatrix}$ by solving (6) and (8).

Step 5. Find $u(t_i)$ and $V(\tau_i)$ by using the formula (5) and (7).

So far we have given the algorithm for the solution of (1)-(4) optimisation problem with the have condition of $F_{\delta_i}^{-1}$ being exist. But in many practical problems this condition does not hold [6]. For this reason it is necessary to give a more general algorithm without using this heavy condition. By which the next part deals with this case.

3. The Case of $F_{\delta_i}^{-1}$ Non – Existence

Suppose that $F_{\delta_i}^{-1}$ is not exist. For this case we can look for $\lambda(t)$ Lagrange Multiplier as

$$\lambda(t) = S(t)x(t) + N(t)\nu + \omega(t) \tag{15}$$

by getting help from being linearity of the equations (6) and (8). Where the matrices $S(t), N(t)$ and $w(t)$ have the forms n×n, m×k and n×1 orderly. These matrices will be obtained in the following.

By comparing (9) and (15) we write,

$$S(\tau_p+0)=0, N(\tau_p+0)=-\Phi_1, \omega(\tau_p+0)=0 \tag{16}$$

and with the help of [3, p.280] we obtain

$$S(\tau_p-0)=0, N(\tau_p-0)=-F_{\delta_p}.\Phi_2', \omega(\tau_p-0)=0 \tag{17}$$

If we put the formula (15) and (6) on the interval $(\tau_{p-1}+0, \tau_p-0)$ we obtain the following matrix differential equations for the matrices of $S(t)$, $N(t)$ and $w(t)$:

$$\dot S(t)=-F'(t)S(t)-S(t)F(t)+S(t)M(t)S(t)-R(t) \tag{18}$$

$$\dot N(t)=(S(t)M(t)-\dot F(t))N(t) \tag{19}$$

$$\omega(t)=(S(t)M(t)-F'(t))\omega(t)-S(t)\upsilon(t) \tag{20}$$

In this equations the conditions are the same as (16), where $M(t)=G(t)C^{-1}(t)G'(t)$.

By the other method [3] we can write

$$-\Phi_2 x(\tau_p+0)=N'x(t)+n(t)\upsilon+w(t) \tag{21}$$

It can be shown easly that the followings are satisfied:

$$\left.\begin{aligned}\dot n(t)&=N'(t)M(t)N(t)\\ n(\tau_p-0)&=-\Phi_2'M_{p-1}\Phi_2\end{aligned}\right\} \quad \text{and} \tag{22}$$

$$\dot W(t)=N'(t)(M(t)w(t)-V(t)) \qquad W(\tau_p-0)=0$$

(23)

By solving the equations (18) – (20) , (22) , (23) under the conditions (17) up to the point $\tau_{p-1}+0$ we can obtain $S(\tau_{p-1}+0)$, $N(\tau_{p-1}+0)$, $\omega(\tau_{p-1}+0)$, $n(\tau_{p-1}+0)$, $w(\tau_{p-1}+0)$.

By taking $\lambda(\tau_{p-1}+0)$and $\lambda(\tau_{p-1}-0)$) as (15) and using (8) we can obtain the following equations for the required τ_i:

$$S(\tau_i-0)=F_{\delta_i}S(\tau_i+0)[E+M_iS(\tau_i+0)]^{-1}F_{\delta_i} \tag{24}$$

$$N(\tau_i-0)=F_{\delta_i}[E+S(\tau_i+0)M_i]^{-1}N(\tau_i+0) \tag{25}$$

$$\omega(\tau_i-0)=F_{\delta_i}[E+S(\tau_i+0)M_i]^{-1}\omega(\tau_i+0)$$

$$n(\tau_i-0)=n(\tau_i+0)-N'(\tau_i+0)[E+M_iS(\tau_i+0)]^{-1}M_iN(\tau_i+0) \tag{27}$$

$$w(\tau_i-0)=W(\tau_i+0)-N'(\tau_i+0)[E+M_iS(\tau_i+0)]^{-1}M_iw(\tau_i+0) \tag{28}$$

Thus if we continue this procedure up to the zero and taking ν as the constant Lagrange Multiplier corresponding to condition (3)we get the following algebraic linear system;

$$\begin{bmatrix} S(0) & N(0)+\Phi_1{}' \\ N'(0)+\Phi_1 & n(0) \end{bmatrix}\begin{bmatrix} x(0) \\ \nu \end{bmatrix}=\begin{bmatrix} -w(0) \\ q-w(0) \end{bmatrix} \tag{29}$$

After finding $x(0)$ and ν from this system we find $x(t)$ and $u(t)$ as follows:

$$\dot{x}(t)=\left(F(t)-G(t)C^{-1}(t)P'(t)-M(t)\right)x(t)-M(t)\left(N(t)\nu+w(t)\right)+\upsilon(t) \tag{30}$$

$$u(t)=-C^{-1}(t)\left[P'(t)+G'(t)S(t)\right]x(t)+G'(t)N(t)\nu+G'(t)w(t) \tag{31}$$

As for the points τ_i we have

$$x(\tau_i+0)=L_1(\tau_i)x(\tau_i-0)-L_2(\tau_i)\nu+L_3(\tau_i) \tag{32}$$

$$V(\tau_i)=-C_i{}^{-1}\Gamma_i\{S(\tau_i+0)L_1(\tau_i)x(\tau_i-0)+N(\tau_i+0)-S(\tau_i+0)L_2(\tau_i)\nu+S(\tau_i+0)L_3(\tau_i)+w(\tau_i+0)\} \tag{33}$$

Where;

$$\begin{aligned} L_1(\tau_i)&=\left(E+M_iS(\tau_i+0)\right)^{-1}F_{\delta_i} \\ L_2(\tau_i)&=\left(E+M_iS(\tau_i+0)\right)^{-1}M_iN(\tau_i+0) \\ L_3(\tau_i)&=-\left(E+M_iS(\tau_i+0)\right)^{-1}M_iw(\tau_i+0) \end{aligned}$$

Finally we can write the following algorithm when $F_{\delta_i}{}^{-1}$ does not exist:

Algorithm 2.

Step 1. Construct the matrices $F(t)$, $G(t)$, $V(t)$, $R(t)$, $S(t)$, F_{δ_i}, Γ_i, C_i, Φ_1 and Φ_2 from (1) - (4).

Step 2. Solve the equations (18) - (23) on the interval $(\tau_i+0,\tau_{i+1}-0)$.

Step3. Solve also $S(\tau_i-0)$, $N(\tau_i-0)$, $\omega(\tau_i-0)$, $n(\tau_i-0)$, $w(\tau_i-0)$ by the equations (24) – (28). Then calculate $S(0)$, $N(0)$, $n(0)$, $\omega(0)$, $w(0)$.

Step 4. Find $x(t)$ and $u(t)$ from (30) and (31) by using the initial conditions $x(0)$. Find also $x(t)$ and $V(t_i)$ from (32) and (33).

The advantage of this algorithm that the continuous and the discrete cases are the special cases and the coefficients matrix of (29) being symmetric. This show that the condition number of the symmetric matrix is one of the good ones as being shown in [1,2].

The following example is an illustration of this algorithm [5].

Example.

Let the system of

$$\dot{x} = Fx + Gu$$

shows the motion of the object, where $x = [\varphi_1 \quad \dot{\varphi}_1 \quad \varphi_2 \quad \dot{\varphi}_2]^T$, $u = [\eta_1 \quad \eta_2]^T$,

$$F = \theta_{01}^{-1}\Omega, \quad G = \theta_{01}^{-1}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \theta_{01} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \theta_{22} & 0 & \theta_{24} \\ 0 & 0 & 1 & 0 \\ 0 & \theta_{42} & 0 & \theta_{44} \end{bmatrix}, \quad \theta_{01}^{-1} = \frac{1}{\beta}\begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \theta_{44} & 0 & -\theta_{24} \\ 0 & 0 & \beta & 0 \\ 0 & -\theta_{42} & 0 & \theta_{22} \end{bmatrix},$$

$$\Omega = g\begin{bmatrix} 0 & 1/g & 0 & 0 \\ w_{21} & 0 & w_{23} & 0 \\ 0 & 0 & 0 & 1/g \\ w_{41} & 0 & w_{43} & 0 \end{bmatrix},$$

$$\beta = \theta_{22}\theta_{44} - \theta_{24}\theta_{42}, \quad \theta_{44} = m_0\left(\tau^2 + \rho^2\right), \quad \theta_{22} = mh^2 + 2m_0\left[(h-\tau)^2 + \rho^2\right],$$

$$\theta_{24} = \theta_{42} = m_0\left[\tau(h-\tau) - \rho^2\right], \quad w_{21} = mh + 2m_0(h-\tau), \quad w_{23} = w_{22} = m_0\tau, \quad w_{43} = -m_0\tau$$

Here we must explain that the interval $(0,\tau)$ is not devided the subintervals and the time $\tau_p = \tau$

$$x(\tau + 0) = F_\delta x(\tau - 0), v(\tau) = 0$$

$$F_\delta = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha = \frac{\theta_{24}}{\theta_{22}} = \frac{m_0\left[\tau(h-\tau) - \rho^2\right]}{mh^2 + 2m_0\left[(h-\tau)^2 + \rho^2\right]}$$

The matrices to corresponding boundary conditions can be obtained from (3) as follows see [3]:

$$\Phi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4/h \\ 0 \end{bmatrix}$$

Moreover m=74,4 kg., m_0=10,2 kg., τ= 0,3m., ρ = 0,24248m., τ =0,8575s. C=$diag\{c_1,c_2\}$, c_1=10, c_2 =1, l=0.694m., h= 1,045m. are also be considered. Under these hypothesis by using the Algorithm 2 we can obtain the following table orbit of the programme trajectory and control for t_i in interval of $(0,\tau)$:

t	0	$\tau/5$	$2\tau/5$	$3\tau/5$	$4\tau/5$	$\tau-0$	$\tau+0$
φ_1	-0.332	-0.162	-0.048	-0.048	0.162	0.332	-0.332
$\dot{\varphi}_1$	1.247	0.786	0.539	0.539	0.780	1.247	1.247
φ_2	-.667	-0.54	-0.218	0.218	0.54	0.664	-0.664
$\dot{\varphi}_2$	0.0	1.415	2.338	2.338	1.415	-0.64×10^{-7}	0.0
ξ_1	0.58×10^{-2}	0.66×10^{-2}	0.28×10^{-2}	-0.28×10^{-2}	-0.66×10^{-2}	-0.58×10^{-2}	
ξ_2	-3.284	-3.256	-1.345	3.289	3.256	3.289	

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