

# CONDITIONING AND SENSITIVITY ANALYSIS OF THE TWO-POINT BOUNDARY VALUE PROBLEM FOR A LINEAR DIFFERENCE EQUATIONS SYSTEM

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**Abstract.** In this paper we present conditioning and sensitivity analysis of the solution of the homogeneous two-point boundary value problems (TPBVP) for the system of the linear difference equations of the form

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi; \quad Rx(n_1) = \psi; \quad \{n : n, n_0, n_1 \in \mathbb{Z}, n_0 \leq n \leq n_1\}, \end{cases}$$

where  $A, L, R$  are respectively  $N \times N$ ,  $k \times N$  and  $(N-k) \times N$  real matrices,  $\varphi, \psi$  and  $f(n)$  are respectively  $N, N-k$  and  $N$  dimensional real column vectors.

**Key words.** Difference equations, two-point boundary value problem, conditioning, sensitivity analysis.

**AMS subject classification.** 39 A 99

## 1. INTRODUCTION.

The theory of difference equations is a lot richer than the corresponding theory of differential equations. Many authors have been studied several problems related to difference equations. Such as stability theory [1], existence and uniqueness theorem [2,3], transmission of information [4], signal processing, oscillation [5], control and dynamic systems [6,7], etc. (see also [8]).

Knowledge about the accuracy in the computation of the solution components is also important [9]. First-order sensitivity analysis involves examination of the effects of differential variations in the fixed coefficients or boundary conditions of a mathematical model. Sensitivity calculations may be required for gradient evaluation in optimizations, in experimental design and analysis, and in many phases of chemical process design [10]. The sensitivity analysis of control problems is current interest in the last fifteen years [11]-[14]. A common task for this analysis is to estimate the perturbation in the solution of a given problem as a function of the perturbations in the data. This is important from both theoretical and practical point of view. In literature perturbation results are available for some basic linear control problems such as computing matrix exponentials, solving linear and quadratic matrix equations (algebraic and differential).

Our aim in this article is to present the notion of conditioning and sensitivity notions for TPBVPs for difference equations.

It is known that for a nonsingular  $N \times N$  matrix  $A$ , the system of difference equations  $x(n+1) = Ax(n)$  has  $N$ -linearly independent solutions. Where

$$x(n) = (x_1(n), x_2(n), \dots, x_N(n))^T$$

is an  $N$  column vector. The matrix with these independent solutions as its columns is called as a fundamental matrix of the system and denoted by  $\Psi(A, n) = (\psi_1, \psi_2, \dots, \psi_N)$  [1,2,6].

## 2. TWO-POINT BOUNDARY VALUE PROBLEMS FOR THE SYSTEM OF LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

We consider the system of difference equations

$$x(n+1) = Ax(n).$$

If for some  $n_0 \geq 0$ ,  $x(n_0) = x_0$  is specified, then the considered system with this condition is called as an initial value problem. If the matrix  $A$  is a  $N \times N$  real nonsingular matrix then this problem has

$$x(n) = A^{n-n_0} x(n_0) \quad (2.1)$$

the unique solution. Now we consider the initial value problem about the system of the nonhomogeneous difference equations

$$x(n+1) = Ax(n) + f(n), \quad x(n_0) = x_0.$$

Therefore the solution of this system is

$$x(n) = A^{n-n_0} x(n_0) + \sum_{k=n_0}^{n-1} A^{n-k-1} f(k) \quad (2.2)$$

[3,15]. We give the following two definitions and a lemma for the sake of use in the sequel:

**Definition 2.1.** Let  $A$   $N \times N$  matrix,  $L$   $k \times N$  matrix and  $R$   $(N-k) \times N$  matrix are real matrices.  $\varphi$ ,  $\psi$  and  $f(n)$  are real column vectors with  $N$ ,  $(N-k)$  and  $N$  elements orderly. Then the following problem is called as a two-point boundary value problem for the system of the linear difference equations with constant coefficients [2]:

$$\begin{cases} x(n+1) = Ax(n) + f(n) \\ Lx(n_0) = \varphi; Rx(n_1) = \psi; \{n : n, n_0, n_1 \in \mathbb{Z}, n_0 \leq n \leq n_1\}, \end{cases} \quad (2.3)$$

**Definition 2.2.** For an  $N \times N$  nonsingular matrix  $A$  the number  $\mu(A) = \|A\| \|A^{-1}\|$  is called as the condition number of  $A$  [9,16]. Where  $\|A\|$  denotes the matrix norm of  $A$  [15].

**Lemma 2.1.** Let  $A$  and  $B$  are real  $N \times N$  matrices and let  $\|B\| < \|A\|$  then the following inequality is satisfied:

$$\|(A+B)^n - A^n\| \leq \frac{\|B\| \|A\|^{(n-1)}}{1 - e^{-\frac{\|B\|}{\|A\|}}} \quad (2.4)$$

**Proof.** To show this inequality we need to consider the following initial-value problem :

$$\begin{cases} X(n+1) = AX(n) + BX(n) \\ X(0) = I \end{cases} \quad (2.5)$$

With the help of (2.1) the unique solution of this problem is

$$X(n) = (A + B)^n \quad (2.6)$$

If we consider this problem as a nonhomogeneous system problem then with the help of (2.2) the unique solution of this problem is

$$X(n) = A^n + \sum_{k=0}^{n-1} A^{n-k-1} B X(k) \quad (2.7)$$

Since the (2.5) problem has only one solution we can write the following equality of

$$(A + B)^n - A^n = \sum_{k=0}^{n-1} A^{n-k-1} B (A + B)^k$$

By using this equality we continue as follows:

$$\begin{aligned} \|(A + B)^n - A^n\| &\leq \|B\| \sum_{k=0}^{n-1} \|A^{n-k-1}\| \|(A + B)^k\| \\ &\leq \|B\| \sum_{k=0}^{n-1} \|A\|^{n-k-1} (\|A\| + \|B\|)^k \\ &\leq \|B\| \sum_{k=0}^{n-1} \|A\|^{n-1} \left(1 + \frac{\|B\|}{\|A\|}\right)^k \\ &\leq \|B\| \|A\|^{n-1} \sum_{k=0}^{n-1} e^{k \frac{\|B\|}{\|A\|}} \\ &\leq \|B\| \|A\|^{n-1} \sum_{k=0}^{\infty} e^{k \frac{\|B\|}{\|A\|}} \\ &= \|B\| \|A\|^{n-1} \frac{1}{1 - e^{-\frac{\|B\|}{\|A\|}}} \\ \|(A + B)^n - A^n\| &\leq \frac{\|B\| \cdot \|A\|^{(n-1)}}{1 - e^{-\frac{\|B\|}{\|A\|}}} \end{aligned}$$

This completes the proof.

## 2.1. Existing and Uniqueness Theorems For The TPBVP

**Theorem 2.1.** We consider the following two-point boundary value problem;

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi; Rx(n_1) = \psi; \{n : n, n_0, n_1 \in Z, n_0 \leq n \leq n_1\}, \end{cases}$$

Then the following two cases can be met:

**Case 1.** If  $A$  is a nonsingular matrix and  $\Psi(A, n)$  is the fundamental matrix of  $x(n+1) = Ax(n)$ ,  $n = 0, \pm 1, \pm 2, \dots$ . Furthermore let  $W$  be given as  $W = \Psi(A, n_1 - n_0) \Psi^{-1}(0)$ .

**Case 2.** If  $A$  is a singular matrix then there exist an orthogonal matrix  $U$  such that

$$U^* A U = \begin{pmatrix} A_1 & B \\ 0 & A_0 \end{pmatrix},$$

is satisfied. Where  $A_0$  is  $N_0 \times N_0$  real matrix having all eigenvalues as 0 and  $\det A_1 \neq 0$ .  $\Psi(A_1, n)$  is the fundamental matrix of  $x(n+1) = A_1 x(n)$ . Let  $W$  be given as

$$W = \Psi(A, n_1 - n_0) (\Psi^{-1}(A_1, 0), 0) U^*.$$

For these two cases we can define the matrix  $H$  as

$$H = \begin{pmatrix} L \\ RW \end{pmatrix}.$$

If the matrix  $H$  is a nonsingular matrix, then the problem has a unique solution [2].

**Proof.**

**Case 1.** By the hypothesis  $A$  is a nonsingular matrix. Since  $\Psi(A, n)$  is the fundamental matrix then the general solution of the given  $x(n+1) = Ax(n)$  difference system can be written as

$$x(n) = \Psi(A, n - m) \Psi^{-1}(A, 0) x(m).$$

Therefore

$$x(n_1) = \Psi(A, n_1 - n_0) \Psi^{-1}(A, 0) x(n_0),$$

is satisfied. For the given two boundary conditions in the theorem

$$Lx(n_0) = \varphi; R\Psi(A, n_1 - n_0) \Psi^{-1}(A, 0) x(n_0) = \psi,$$

should be verified. So by using  $W = \Psi(A, n_1 - n_0) \Psi^{-1}(A, 0)$  we can see that  $\det H \neq 0$  is true. Therefore the problem has a unique solution in this case.

**Case 2.** By the hypothesis,  $A$  is a singular matrix. Therefore the general solution of the difference system can be written as

$$x(n) = \Psi(A, n - m) (\Psi^{-1}(A_1, 0), 0) U^* x(m).$$

So

$$x(n_1) = \Psi(A, n_1 - n_0) (\Psi^{-1}(A_1, 0), 0) U^* x(n_0),$$

is satisfied. We also know that

$$Lx(n_0) = \varphi; R\Psi(A, n_1 - n_0) (\Psi^{-1}(A_1, 0), 0) U^* x(n_0) = \psi.$$

must hold. By using

$$W = \Psi(A, n_1 - n_0) (\Psi^{-1}(A_1, 0), 0) U^*$$

we can see that  $\det H \neq 0$  is satisfied. So the problem has a unique solution for this case.

It is clear that the solution of the TPBVP is  $x(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \leq n \leq n_1$ .

**2.2. Conditioning Of TPBVP**

With the help of the solution  $x(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \leq n \leq n_1$  we may write the following linear algebraic equation:

$$H.A^{-(n-n_0)} x(n) = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \leq n \leq n_1$$

We can define the condition number of the TPBVP as

$$\mu(HA^{-(n-n_0)}) = \|HA^{-(n-n_0)}\| \cdot \|(HA^{-(n-n_0)})^{-1}\|.$$

### 3. ON THE SENSITIVITY OF TPBVP

In this part, we examine the behavior of the solution of given any TPBVP according to the small changes of initial values. Therefore, here we examine the sensitivity property of a TPBVP.

**Theorem 3.1.** We consider the following two TPBVP's :

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi ; Rx(n_1) = \psi ; \{n : n, n_0, n_1 \in Z, n_0 \leq n \leq n_1\}, \end{cases} \quad (3.1)$$

Suppose the following problem is given

$$\begin{cases} y(n+1) = Ay(n) \\ Ly(n_0) = \tilde{\varphi} ; Ry(n_1) = \tilde{\psi} ; \{n : n, n_0, n_1 \in Z, n_0 \leq n \leq n_1\}, \end{cases} \quad (3.2)$$

Then, for the problems (3.1) and (3.2)

$$\frac{\|x(n) - y(n)\|}{\|x(n)\|} \leq \mu(A^{n-n_0} H^{-1}) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|}, n_0 \leq n \leq n_1 \quad (3.3)$$

is satisfied. Where A is a nonsingular matrix.

**Proof.** Since A is a nonsingular matrix then the solutions of (3.1) and (3.2) can be written orderly as

$$\begin{aligned} x(n) &= A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \leq n \leq n_1, \\ y(n) &= A^{n-n_0} H^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}, n_0 \leq n \leq n_1. \end{aligned}$$

Here the matrix H has the following form

$$H = \begin{pmatrix} L \\ RW \end{pmatrix} = \begin{pmatrix} L \\ RA^{n_1-n_0} \end{pmatrix}.$$

Therefore, by the hypothesis we can see that the following is satisfied:

$$x(n) - y(n) = A^{n-n_0} H^{-1} \left[ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right], n_0 \leq n \leq n_1 \quad (3.4)$$

$$\begin{aligned} \|x(n) - y(n)\| &\leq \|A^{n-n_0} H^{-1}\| \left\| (A^{n-n_0} H^{-1})^{-1} \left[ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right] \right\| \|A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\| \\ &\leq \mu(A^{n-n_0} H^{-1}) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|} \|x(n)\| \end{aligned}$$

$$\|x(n) - y(n)\| \leq \mu(A^{n-n_0} H^{-1}) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|} \|x(n)\|, n_0 \leq n \leq n_1$$

$$\frac{\|x(n) - y(n)\|}{\|x(n)\|} \leq \mu(A^{n-n_0} H^{-1}) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|}, n_0 \leq n \leq n_1 .$$

If we examine the upper bound of this relative error we get the result that the condition number  $\mu(A^{n-n_0} H^{-1})$  is effective. Therefore as the condition number  $\mu$  gets bigger then the relative error also gets bigger. By looking at the above upper bound we see that this holds locally. In the above inequality if the upper bound does not depend on the number  $n$  then it will be the global upper bound for the relative error.

Now we give the following theorem, which deals with the sensitivity of the problem, only when the elements of  $L$  and  $R$  have small changes.

**Theorem 3.2.** Let the following problems are given:

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi ; Rx(n_1) = \psi ; \{n : n, n_0, n_1 \in Z, n_0 \leq n \leq n_1\}, \end{cases} \quad (3.5)$$

$$\begin{cases} y(n+1) = Ay(n) \\ \tilde{L}y(n_0) = \varphi ; \tilde{R}y(n_1) = \psi ; \{n : n, n_0, n_1 \in Z, n_0 \leq n \leq n_1\}, \end{cases} \quad (3.6)$$

Then for these two problems the inequality of

$$\frac{\|x(n) - y(n)\|}{\|x(n)\|} \leq \mu(A^{n-n_0} H^{-1}) \left( \mu(H) \frac{\|L - \tilde{L}\|}{\|L\|} + \frac{\|R - \tilde{R}\|}{\|R\|} \mu(A^{n_1-n_0}) \right) \frac{1}{1 - \mu(H) \frac{\|L - \tilde{L}\|}{\|L\|} + \frac{\|R - \tilde{R}\|}{\|R\|} \mu(A^{n_1-n_0})} \quad (3.7)$$

is satisfied.

**Proof.** As it can be obtained easily the solutions of these problems are

$$x(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \leq n \leq n_1,$$

$$y(n) = A^{n-n_0} \tilde{H}^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \leq n \leq n_1$$

orderly. By using the hypotheses and these solutions we can see that the below equalities and inequalities are satisfied:

$$x(n) - y(n) = -A^{n-n_0} (\tilde{H}^{-1} - H^{-1}) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \quad (3.8)$$

$$= -A^{n-n_0} (\tilde{H}^{-1} - H^{-1}) (A^{n-n_0} H^{-1})^{-1} x(n)$$

$$\frac{\|x(n) - y(n)\|}{\|x(n)\|} \leq \|A^{n-n_0} (\tilde{H}^{-1} - H^{-1})\| \|(A^{n-n_0} H^{-1})^{-1}\|$$

$$\leq \|A^{n-n_0} H^{-1}\| \|H(\tilde{H}^{-1} - H^{-1})\| \|(A^{n-n_0} H^{-1})^{-1}\|$$

$$= \mu(A^{n-n_0} H^{-1}) \|H(\tilde{H}^{-1} - H^{-1})\|$$

$$\begin{aligned}
H(\tilde{H}^{-1} - H^{-1}) &= -(\tilde{H} - H)\tilde{H}^{-1} \\
\|H(\tilde{H}^{-1} - H^{-1})\| &\leq \|\tilde{H} - H\| \|\tilde{H}^{-1}\| \\
&= \|\tilde{H} - H\| \frac{1}{\sigma_1(\tilde{H})}
\end{aligned}$$

Here  $\sigma_1(\tilde{H})$  is the least singular values of the matrix  $\tilde{H}$ .

$$\begin{aligned}
\|H(\tilde{H}^{-1} - H^{-1})\| &\leq \|\tilde{H} - H\| \frac{1}{\sigma_1(H) - \|H - \tilde{H}\|} \\
&= \frac{1}{\|H\|} \|\tilde{H} - H\| \frac{1}{\frac{\sigma_1(H)}{\|H\|} - \frac{\|H - \tilde{H}\|}{\|H\|}} \\
&= \mu(H) \frac{\|\tilde{H} - H\|}{\|H\|} \frac{1}{1 - \mu(H) \frac{\|H - \tilde{H}\|}{\|H\|}} \\
\frac{\|x(n) - y(n)\|}{\|x(n)\|} &\leq \mu(A^{n-n_0} H^{-1}) \mu(H) \frac{\|\tilde{H} - H\|}{\|H\|} \frac{1}{1 - \mu(H) \frac{\|H - \tilde{H}\|}{\|H\|}} \tag{3.9}
\end{aligned}$$

$$H - \tilde{H} = \begin{pmatrix} L - \tilde{L} & \\ (R - \tilde{R}) & A^{n_1-n_0} \end{pmatrix}$$

$$\begin{aligned}
\|H - \tilde{H}\| &\leq \left\| \begin{pmatrix} L - \tilde{L} \\ 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 \\ (R - \tilde{R}) & A^{n_1-n_0} \end{pmatrix} \right\| \\
&\leq \|L - \tilde{L}\| + \|(R - \tilde{R})A^{n_1-n_0}\| \\
\frac{\|H - \tilde{H}\|}{\|H\|} &\leq \frac{\|L - \tilde{L}\|}{\|H\|} + \frac{\|R - \tilde{R}\|}{\|H\|} \|A^{n_1-n_0}\| \\
&\leq \frac{\|L - \tilde{L}\|}{\|L\|} + \frac{\|R - \tilde{R}\|}{\|RA^{n_1-n_0}\|} \|A^{n_1-n_0}\| \\
&\leq \frac{\|L - \tilde{L}\|}{\|L\|} + \frac{\|R - \tilde{R}\|}{\|R\|} \frac{\|A^{n_1-n_0}\|}{\sigma_1(A^{n_1-n_0})} \\
&\leq \frac{\|L - \tilde{L}\|}{\|L\|} + \frac{\|R - \tilde{R}\|}{\|R\|} \mu(A^{n_1-n_0})
\end{aligned}$$

$$\frac{\|x(n) - y(n)\|}{\|x(n)\|} \leq \mu(A^{n-n_0} H^{-1}) \left( \mu(H) \frac{\|L - \tilde{L}\|}{\|L\|} + \frac{\|R - \tilde{R}\|}{\|R\|} \mu(A^{n_1-n_0}) \right) \frac{1}{1 - \mu(H) \frac{\|L - \tilde{L}\|}{\|L\|} + \frac{\|R - \tilde{R}\|}{\|R\|} \mu(A^{n_1-n_0})}$$

Thus, the proof of the theorem is completed.

Now, we have arrived at the point that when the elements of  $A, L, R, \varphi, \psi$  have too small changes and the behavior of the solutions of some TPBVPs regarding with these changes. To do these, we need the following theorem:

**Theorem 3.3.**

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi ; Rx(n_1) = \psi ; \{n : n, n_0, n_1 \in Z, n_0 \leq n \leq n_1\}, \end{cases} \quad (3.10)$$

$$\begin{cases} y(n+1) = \tilde{A}y(n) \\ \tilde{L}y(n_0) = \tilde{\varphi} ; \tilde{R}y(n_1) = \tilde{\psi} ; \{n : n, n_0, n_1 \in Z, n_0 \leq n \leq n_1\}, \end{cases} \quad (3.11)$$

Suppose the problems (3.10) and (3.11) be considered. Where, as we mentioned above,  $\tilde{A}, \tilde{L}, \tilde{R}, \tilde{\varphi}, \tilde{\psi}$  and  $A, L, R, \varphi, \psi$  are too close to each other in the sense of norm respectively. In addition to these the inequality of  $\|R - \tilde{R}\| \leq (0.1)\|R\|$  be at hand. The closeness of these solutions depends on the condition number  $\mu(H) = \|H\| \|H^{-1}\|$ . Let

$$S_1 = \|x(n)\| \mu(A^{n-n_0} H^{-1}) \mu(H) \left[ \|L - \tilde{L}\| + \|R - \tilde{R}\| \|A^{n_1-n_0}\| + (1.1) \|R\| \frac{\|A - \tilde{A}\| \|A\|^{n_1-n_0-1}}{1 - e^{-\frac{\|A-\tilde{A}\|}{\|A\|}}} \right]$$

$$\cdot \frac{1}{\|H\| - \mu(H) \left[ \|L - \tilde{L}\| + \|R - \tilde{R}\| \|A^{n_1-n_0}\| + (1.1) \|R\| \frac{\|A - \tilde{A}\| \|A\|^{n_1-n_0-1}}{1 - e^{-\frac{\|A-\tilde{A}\|}{\|A\|}}} \right]}$$

$$S_2 = \mu(A^{n-n_0} \tilde{H}^{-1}) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|} \|x(n)\|, \quad S_3 = \frac{\|A - \tilde{A}\| \|A\|^{n-n_0-1}}{1 - e^{-\frac{\|A-\tilde{A}\|}{\|A\|}}} \|\tilde{H}^{-1}\| \left\| \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|.$$

Then the inequality of

$$\|y(n) - x(n)\| \leq S_1 + S_2 + S_3 \quad (3.12)$$

holds.

**Proof.** As it can be obtained easily that the solutions of the problems (3.10) and (3.11) are

$$x(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \leq n \leq n_1, \quad H = \begin{pmatrix} L \\ RA^{n_1-n_0} \end{pmatrix},$$

$$y(n) = \tilde{A}^{n-n_0} \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}, n_0 \leq n \leq n_1, \quad \tilde{H} = \begin{pmatrix} \tilde{L} \\ \tilde{R} \tilde{A}^{n_1-n_0} \end{pmatrix},$$

orderly. By using them we can see that the following equalities are satisfied:

$$\begin{aligned} x(n) - y(n) &= A^{n-n_0} \left[ H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right] + [A^{n-n_0} - \tilde{A}^{n-n_0}] \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \\ &= A^{n-n_0} \left[ H^{-1} - \tilde{H}^{-1} \right] \begin{pmatrix} \varphi \\ \psi \end{pmatrix} + A^{n-n_0} \tilde{H}^{-1} \left[ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right] + [A^{n-n_0} - \tilde{A}^{n-n_0}] \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}. \end{aligned}$$

For the sake of shortness let us write the following.

$$C_1 = A^{n-n_0} \left[ H^{-1} - \tilde{H}^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right], C_2 = A^{n-n_0} \tilde{H}^{-1} \left[ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right], C_3 = \left[ A^{n-n_0} - \tilde{A}^{n-n_0} \right] \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$$

So, the following inequality can be written:

$$\|x(n) - y(n)\| \leq \|C_1\| + \|C_2\| + \|C_3\|.$$

With the help of the Lemma 2.1, the Theorem 3.1 and finally the Theorem 3.2 the followings are satisfied, orderly.

$$\begin{aligned} \|C_3\| &\leq \|A^{n-n_0} - \tilde{A}^{n-n_0}\| \|\tilde{H}^{-1}\| \left\| \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\| \\ \|C_3\| &\leq \frac{\|A - \tilde{A}\| \|A\|^{n-n_0-1}}{1 - e^{-\|A\|}} \|\tilde{H}^{-1}\| \left\| \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\| \end{aligned} \quad (3.13)$$

$$\|C_2\| \leq \mu(A^{n-n_0} \tilde{H}^{-1}) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|} \|x(n)\| \quad (3.14)$$

$$\|C_1\| \leq \|x(n)\| \mu(A^{n-n_0} H^{-1}) \mu(H) \|\tilde{H} - H\| \frac{1}{\|\tilde{H}\| - \mu(H) \|\tilde{H} - H\|} \quad (3.15)$$

By the hypotheses on H and  $\tilde{H}$  we can write the following inequality

$$\begin{aligned} \|H - \tilde{H}\| &\leq \left\| \begin{pmatrix} L - \tilde{L} \\ 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 \\ RA^{n_1-n_0} - \tilde{R}\tilde{A}^{n_1-n_0} \end{pmatrix} \right\| \leq \|L - \tilde{L}\| + \|RA^{n_1-n_0} - \tilde{R}\tilde{A}^{n_1-n_0}\| \\ \|H - \tilde{H}\| &\leq \|L - \tilde{L}\| + \|(R - \tilde{R})A^{n_1-n_0} + \tilde{R}(A^{n_1-n_0} - \tilde{A}^{n_1-n_0})\| \\ &\leq \|L - \tilde{L}\| + \|R - \tilde{R}\| \|A^{n_1-n_0}\| + \|\tilde{R}\| \|A^{n_1-n_0} - \tilde{A}^{n_1-n_0}\| \end{aligned}$$

is obtained easily. By the Lemma 2.1 that the last inequality takes the following form:

$$\|H - \tilde{H}\| \leq \|L - \tilde{L}\| + \|R - \tilde{R}\| \|A^{n_1-n_0}\| + [\|R\| + \|R - \tilde{R}\|] \frac{\|A - \tilde{A}\| \|A\|^{n_1-n_0-1}}{1 - e^{-\|A\|}}.$$

Thus, by using the results obtained above and adding the hypotheses of  $\|R - \tilde{R}\| \leq (0.1)\|R\|$  to these we can see that the inequality (3.15) takes the following form

$$\begin{aligned} C_1 &\leq \|x(n)\| \mu(A^{n-n_0} H^{-1}) \mu(H) \left[ \|L - \tilde{L}\| + \|R - \tilde{R}\| \|A^{n_1-n_0}\| + (1.1)\|R\| \frac{\|A - \tilde{A}\| \|A\|^{n_1-n_0-1}}{1 - e^{-\|A\|}} \right] \\ &\quad \cdot \frac{1}{\|\tilde{H}\| - \mu(H) \left[ \|L - \tilde{L}\| + \|R - \tilde{R}\| \|A^{n_1-n_0}\| + (1.1)\|R\| \frac{\|A - \tilde{A}\| \|A\|^{n_1-n_0-1}}{1 - e^{-\|A\|}} \right]} \end{aligned}$$

So, this completes the proof.

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