CONDITIONING AND SENSITIVITY ANALYSIS OF THE TWO-POINT BOUNDARY VALUE PROBLEM FOR THE SYSTEM OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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ABSTRACT

In this study, the sensitivity of two-point boundary value problem is investigated for a system of homogeneous, linear, ordinary differential equations with constant coefficients :

$$\frac{d}{dt}x(t) = Ax(t)\ , \quad t \in [t_0, t_1]$$

$$Lx(t_0) = \varphi, \quad Rx(t_1) = \psi$$

Key words: Linear system, Condition number, Sensitivity.

1. INTRODUCTION

Huge developments in calculation techniques have been achieved in recent years. Main subject among these developments is to examine the closeness of the numerical solutions to the exact solution [1-4]. Knowledge about the accuracy in the computation of the solution components is important [5]. First-order sensitivity analysis involves examination of the effects of differential variations in the fixed coefficients or boundary conditions of a mathematical model. Sensitivity calculations may be required for gradient evaluation in optimizations, in experimental design and analysis, and in many phases of chemical process design [6]. The sensitivity analysis of control problems is current interest in the last fifteen years [7-10]. A common task for this analysis is to estimate the perturbation in the solution of a given problem as a function of the perturbations in the data. This is important from both theoretical and practical point of views. Perturbation results are available for some basic linear control problems such as computing matrix exponentials, solving linear and quadratic matrix equations (algebric and differential). Our aim in this article is to present sensitivity notion for the TPBVP.

In our work, the effects of the accuracy of the set of numbers used by the computer, the errors in the data and ill conditionedness of the problem on the solutions of the two-point boundary value problems are underlined.

Up to this time, TPBVP for systems of differential equations has been studied in the sense of existence and uniqueness theorem [11], the synthesis of optimal filtering algorithms [12,13], wave propagation [14] etc. Several computational methods have also been developed for solving TPBVP's [15-17].

2. *THE TWO POINT BOUNDARY VALUE PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS*

Definition 1. The problem of finding the solution of the problem

$$\frac{d}{dt}x(t) = Ax(t)\,, \quad t \in [t_0, t_1]$$
$$Lx(t_0) = \varphi, \quad Rx(t_1) = \psi \tag{1}$$

is called 'two-point boundary value problem for a homogeneous, linear, ordinary differential equation with constant coefficients'. Where A, R, L are N×N, k×N, (N-k)×N type real matrices respectively. φ is a vector with k components, ψ is a vector with (N-k) components and x(t) is N dimensional unknown vector function on the given interval.

Definition 2 (Fundamental Matrix). Let N dimensional vector $\{\psi_1(t), \psi_2(t),...,\psi_N(t)\}$ be the basis of vector space of the functions of solutions of the system given in (1).

Then, the matrix

$$\psi(t) = [\ \psi_1(t), \psi_2(t),..., \psi_N(t)\ ]$$

is called a fundamental matrix of the system given in (1).

Namely, if $\psi(t)$ is a fundamental matrix, then one has [1]

$$\frac{d}{dt}\psi(t) = A\psi(t); \quad \det \psi(t) \neq 0\,.$$

The only solution of the Cauchy problem

$$\frac{d}{dt}X(t) = A\,X(t)$$
$$X(0) = I$$

Is $X(t) = e^{tA}$ which is an exponential matrix function. So, e^{tA} is a fundamental matrix for the given system [18].

Theorem 1. Let us consider the problem (1). If $\Phi(t)$ is a fundamental matrix of the system

$$\frac{d}{dt}x(t) = Ax(t)\,, \quad t \in [t_0, t_1]$$

and if

$$H = \begin{pmatrix} L \\ R\Phi(t_1 - t_0)\Phi^{-1}(0) \end{pmatrix}, \tag{2}$$

then the unique solution of the problem (1) is [9]

$$x(t) = e^{(t-t_0)A} H^{-1} \begin{bmatrix} \varphi \\ \psi \end{bmatrix}. \qquad (3)$$

Note: Let T, t be any numbers. Then for any fundamental matrix Φ(t) satisfies

$$\Phi(T-t)\,\Phi^{-1}(0) = e^{(T-t)A}.$$

Therefore, the matrix (2) is of the form

$$H = \begin{bmatrix} L \\ \mathrm{Re}^{(t_1-t_0)A} \end{bmatrix}$$

and independent of the choice of the fundamental matrix.

2.1. Conditioning Of TPBVP

We consider the following TPBVP:

$$\frac{d}{dt}x(t) = Ax(t), \quad t \in [t_0, t_1]$$

$$Lx(t_0) = \varphi, Rx(t_1) = \psi$$

It is known that any solution of the system

$$\frac{d}{dt}x(t) = Ax(t)$$

with the initial value $x(t_0)$ is

$$x(t) = e^{(t-t_0)A}x(t_0).$$

If we apply the boundary conditions of TPBVP, we obtain the following equation:

$$\begin{bmatrix} L \\ Re^{(t_1-t_0)A} \end{bmatrix} x(t_0) = \begin{bmatrix} \varphi \\ \psi \end{bmatrix}.$$

We know already that $H = \begin{bmatrix} L \\ Re^{(t_1-t_0)A} \end{bmatrix}$. Therefore the solution of this TPBVP is

$$x(t) = e^{(t-t_0)A} H^{-1} \begin{bmatrix} \varphi \\ \psi \end{bmatrix}$$

Hence we may write this solution as linear algebraic form

$$H.e^{-(t-t_0)A}x(t) = \begin{bmatrix} \varphi \\ \psi \end{bmatrix}.$$

It is possible to present the conditioning of the TPBVP by the help of the above system of linear algebraic equations. The condition number of the above TPBVP is defined as

$$\mu\left(He^{-(t-t_0)A}\right) = \left\|He^{-(t-t_0)A}\right\| . \left\|\left(He^{-(t-t_0)A}\right)^{-1}\right\|.$$

Where the norm || || is a spectral norm [4].

3. SENSITIVITY ANALYSIS OF THE SOLUTION

In this part, the effect of the errors in the input data to the solution of the problem (1) is going to be studied. In the following theorems these effects are considered as the sensitivity of the solution of these problems and we determine an upper bound for the absolute errors that have been made.

Theorem 2. Let A be a non singular matrix and the following two point boundary value problem be given:

$$\frac{d}{dt}X(\mathrm{t}) = \mathrm{A}X(\mathrm{t})\,, \quad t \in [t_0, t_1]$$

$$LX(\mathrm{t}_0) = \varphi, \quad RX(\mathrm{t}_1) = \psi \tag{4}$$

and

$$\frac{d}{dt}Y(\mathrm{t}) = A\,Y(\mathrm{t})\,, \quad t \in [t_0, t_1]$$

$$\mathrm{L}\,\mathrm{Y}(\mathrm{t}_0) = \tilde{\varphi}\,, \quad \mathrm{R}\,\mathrm{Y}(\mathrm{t}_1) = \tilde{\psi}\,. \tag{5}$$

Then the corresponding solutions o the two TPBVP's (4) and (5) satisfy the following inequality:

$$\frac{\|X(t)-Y(t)\|}{\|X(t)\|} \le \mu\left(e^{(t-t_0)A}H^{-1}\right)\frac{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\|}{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|} \tag{6}$$

for small $|t_0\ , t_1|$.

Proof. The solutions of the given problems are:

$$X(t) = e^{(t-t_0)A}H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

and

$$Y(t) = e^{(t-t_0)A}H^{-1}\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}.$$

Respectively. Hence we have:

$$\|X(t) - Y(t)\| \le \left\| e^{(t-t_0)A}H^{-1}\left\{\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\}\right\|$$

$$\le \mu\left(e^{(t-t_0)A}H^{-1}\right)\frac{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\|}{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|}\left\| e^{(t-t_0)A}H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|$$

$$\leq \mu\left(e^{(t-t_0)A}H^{-1}\right)\frac{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}-\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\|}{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|}\ \|X(t)\| ,$$

which leads us to inequality (6).

With this theorem, an upper bound has been determined for the relative error of the problems (4) and (5). It is seen that the condition number $\mu\left(e^{(t-t_0)A}H^{-1}\right)$ has an effective role on the upper bound.

Theorem 3. Let A be a nonsingular matrix. Consider the following two-point boundary value problems on the interval of $[t_0,t_1]$:

$$\frac{d}{dt}X(\mathrm{t}) = \mathrm{A}X(\mathrm{t})$$

$$LX(\mathrm{t}_0) =\varphi, \quad RX(\mathrm{t}_1)= \psi \qquad (7)$$

and

$$\frac{d}{dt}Y(\mathrm{t}) = A\,Y(\mathrm{t})$$

$$\tilde{L}\,Y(\mathrm{t}_0) =\varphi, \quad \tilde{R}\,Y(\mathrm{t}_1)= \psi. \qquad (8)$$

Then the corresponding solutions o the above two TPBVP's (7) and (8) satisfy the following inequality:

$$\frac{\|X(t)-Y(t)\|}{\|X(t)\|} \leq \mu\left(e^{(t-t_0)A}H^{-1}\right)\mu(H)\frac{\|H-\tilde{H}\|}{\|H\|}\frac{1}{1-\mu(H)\dfrac{\|H-\tilde{H}\|}{\|H\|}} . \qquad (9)$$

Proof. It is known that the solutions of the given problems are

$$X(t) = e^{(t-t_0)A}H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

and

$$Y(t) = e^{(t-t_0)A}\,\tilde{H}^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}.$$

Respectively. Here H and $\tilde{H}$ are given by:

$$H=\begin{pmatrix}L\\ \mathrm{R}e^{(t_1-t_0)A}\end{pmatrix},\ \tilde{H}=\begin{pmatrix}\tilde{L}\\ \tilde{\mathrm{R}}e^{(t_1-t_0)A}\end{pmatrix}.$$

Then,

$$X(t)- Y(t) = e^{(t-t_0)A}H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - e^{(t-t_0)A}\,\tilde{H}^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

$$= -\, e^{(t-t_0)A}\,(\widetilde{H}^{-1} - H^{-1})\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

$$= -\, e^{(t-t_0)A}\,(\widetilde{H}^{-1} - H^{-1})\,(e^{(t-t_0)A}H^{-1})^{-1}\,(e^{(t-t_0)A}H^{-1})\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

is obtained. Hence we get

$$X(t) - Y(t) = -\, e^{(t-t_0)A}\,(\widetilde{H}^{-1} - H^{-1})\,(e^{(t-t_0)A}H^{-1})^{-1}X(t)\,. \tag{10}$$

Hence for the relative error we have the following upper bound:

$$\frac{\|X(t)-Y(t)\|}{\|X(t)\|} \le \|\, e^{(t-t_0)A}\,(\widetilde{H}^{-1} - H^{-1})\,\|.\|\,(e^{(t-t_0)A}H^{-1})^{-1}\|$$

$$= \|\, e^{(t-t_0)A}H^{-1}H\,(\widetilde{H}^{-1} - H^{-1})\|.\|\,(e^{(t-t_0)A}H^{-1})^{-1}\,\|$$

$$\le \|\, e^{(t-t_0)A}H^{-1}\|.\|\,H\,(\widetilde{H}^{-1} - H^{-1})\|.\|\,(e^{(t-t_0)A}H^{-1})^{-1}\,\|\,,$$

which implies,

$$\frac{\|X(t)-Y(t)\|}{\|X(t)\|} \le \mu\left(e^{(t-t_0)A}H^{-1}\right)\|\,H\,(\widetilde{H}^{-1} - H^{-1})\|\,. \tag{11}$$

Furthermore, by taking the norms of both sides of

$$H\,(\widetilde{H}^{-1} - H^{-1}) = (\widetilde{H} - H)\,\widetilde{H}^{-1}$$

One obtains

$$\|H\,(\widetilde{H}^{-1} - H^{-1})\| \le \|\widetilde{H} - H\|.\|\widetilde{H}^{-1}\|$$

$$= \|\widetilde{H} - H\|\,\frac{1}{\sigma_1(\widetilde{H})}$$

Here $\sigma_1(\widetilde{H})$ are the least singular values of the matrix $\widetilde{H}$.

$$\|\,H(\widetilde{H}^{-1} - H^{-1})\,\| \le \|\,\widetilde{H} - H\,\|\frac{1}{\sigma_1(H) - \|H - \widetilde{H}\|}$$

$$= \frac{1}{\|H\|}\|\,\widetilde{H} - H\,\|\frac{1}{\frac{\sigma_1(H)}{\|H\|} - \frac{\|H - \widetilde{H}\|}{\|H\|}}$$

$$= \mu(H)\frac{\|\widetilde{H} - H\|}{\|H\|}\,\frac{1}{1 - \mu(H)\frac{\|H - \widetilde{H}\|}{\|H\|}}$$

If we substitute this result in (11), the required result (9) is obtained. This completes the proof of the theorem.

Theorem 4. Let us consider the following two problems:

$$\frac{d}{dt}X(t) = AX(t)\,, \quad t\in[t_0, t_1]$$

$$LX(t_0) = \varphi, \quad RX(t_1) = \psi \tag{12}$$

and

$$\frac{d}{dt}Y(t) = \widetilde{A}\ Y(t)\ , \quad t\in[t_0, t_1]$$
$$\widetilde{L}\ Y(t_0) = \widetilde{\varphi}\ , \quad \widetilde{R}\ Y(t_1) = \widetilde{\psi} \tag{13}$$

Where $\widetilde{A},A;\ \widetilde{L},L;\ \widetilde{R},R;\ \widetilde{\varphi},\varphi;\ \widetilde{\psi},\psi$ are couples of matrices and vectors that differ slightly in the sense of the norm $\|\ \|$. Then one has

$$\| X(t) - Y(t) \| \le \| X(t) \|\ \mu\left(e^{(t-t_0)A}H^{-1}\right)\mu(H)\ \|H - \widetilde{H}\| \frac{1}{\| H \| - \mu(H)\| H - \widetilde{H} \|}$$
$$+ \| X(t) \|\ \mu\left(e^{(t-t_0)A}\widetilde{H}^{-1}\right) \frac{\left\| \begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix} \right\|}{\left\| \begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix} \right\|} + \|A - \widetilde{A}\|\ e^{T\|A\|}\ e^{T\|\widetilde{A}\|} \|\widetilde{H}^{-1}\| \left\| \begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix} \right\|.$$

Where

$$H = \begin{pmatrix} L \\ \mathrm{Re}^{TA} \end{pmatrix}. \tag{14}$$

Proof. The solutions of these problems are respectively,

$$X(t) = e^{(t-t_0)A}H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

and

$$Y(t) = e^{(t-t_0)\widetilde{A}}\ \widetilde{H}^{-1}\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix}.$$

Hence one has

$$X(t) - Y(t) = e^{(t-t_0)A}H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - e^{(t-t_0)\widetilde{A}}\ \widetilde{H}^{-1}\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix}$$
$$= e^{(t-t_0)A}H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - e^{(t-t_0)A}\ \widetilde{H}^{-1}\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix} + e^{(t-t_0)A}\ \widetilde{H}^{-1}\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix} - e^{(t-t_0)\widetilde{A}}\ \widetilde{H}^{-1}\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix}$$
$$= e^{(t-t_0)A}\left(H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \widetilde{H}^{-1}\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix}\right) + \left(e^{(t-t_0)A} - e^{(t-t_0)\widetilde{A}}\right)\widetilde{H}^{-1}\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix}$$
$$= e^{(t-t_0)A}\left(H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \widetilde{H}^{-1}\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix} + \widetilde{H}^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \widetilde{H}^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right) + \left(e^{(t-t_0)A} - e^{(t-t_0)\widetilde{A}}\right)\widetilde{H}^{-1}\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix}$$
$$= e^{(t-t_0)A}\left[H^{-1} - \widetilde{H}^{-1}\right]\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + e^{(t-t_0)A}\ \widetilde{H}^{-1}\left[\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix}\right] + \left[e^{(t-t_0)A} - e^{(t-t_0)\widetilde{A}}\right]\widetilde{H}^{-1}\begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix}.$$

Let

$$C_1 = e^{(t-t_0)A}\left(H^{-1} - \widetilde{H}^{-1}\right)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$
$$C_2 = e^{(t-t_0)A}\ \widetilde{H}^{-1}\left(\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}\widetilde{\varphi}\\ \widetilde{\psi}\end{pmatrix}\right)$$

and

$$C_3 = \left(e^{(t-t_0)A} - e^{(t-t_0)\widetilde{A}} \right) \widetilde{H}^{-1} \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix}.$$

Then we have,

$$\| X(t) - Y(t) \| \le \| C_1 \| + \| C_2 \| + \| C_3 \|$$

Using the results obtained in Theorem 2 and 3, we have the following bounds for C_1 and C_2 :

$$\|C_1\| \le \| X(t) \| . \mu(e^{(t-t_0)A} H^{1}) \, \mu(H) \, \|H - \widetilde{H}\| \frac{1}{\| H \| - \mu(H) \, \| H - \widetilde{H} \|}$$

and

$$\| C_2 \| \le \| X(t) \| \, \mu\left(e^{(t-t_0)A} \widetilde{H}^{-1}\right) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right\|}.$$

It is clear that for C_3, we also have the following bound:

$$\| C_3 \| \le \| e^{(t-t_0)A} - e^{(t_1-t_0)\widetilde{A}} \| \, \| \widetilde{H}^{-1} \| \left\| \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right\|$$

$$\le \|A - \widetilde{A}\| \, e^{(t-t_0)\|A\|} \, e^{(t_1-t_0)\|\widetilde{A}\|} \, \| \widetilde{H}^{-1} \| . \left\| \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right\|$$

This completes the proof of the theorem.

From the above result we conclude that the closeness of the solutions depends o the condition number $\mu(H) = \|H\| . \|H^{-1}\|$ and the length $T = t_1 - t_0$ of the time interval.

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