

Chapter 08.05

On Solving Higher Order Equations for Ordinary Differential Equations

After reading this chapter, you should be able to:

1. solve higher order and coupled differential equations,

We have learned Euler's and Runge-Kutta methods to solve first order ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0 \quad (1)$$

What do we do to solve simultaneous (coupled) differential equations, or differential equations that are higher than first order? For example an n^{th} order differential equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = f(x) \quad (2)$$

with $n - 1$ initial conditions can be solved by assuming

$$y = z_1 \quad (3.1)$$

$$\frac{dy}{dx} = \frac{dz_1}{dx} = z_2 \quad (3.2)$$

$$\frac{d^2 y}{dx^2} = \frac{dz_2}{dx} = z_3 \quad (3.3)$$

⋮

$$\frac{d^{n-1} y}{dx^{n-1}} = \frac{dz_{n-1}}{dx} = z_n \quad (3.n)$$

$$\begin{aligned} \frac{d^n y}{dx^n} &= \frac{dz_n}{dx} \\ &= \frac{1}{a_n} \left(-a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} \dots - a_1 \frac{dy}{dx} - a_0 y + f(x) \right) \\ &= \frac{1}{a_n} (-a_{n-1} z_n \dots - a_1 z_2 - a_0 z_1 + f(x)) \end{aligned} \quad (3.n+1)$$

The above Equations from (3.1) to (3.n+1) represent n first order differential equations as follows

$$\frac{dz_1}{dx} = z_2 = f_1(z_1, z_2, \dots, x) \quad (4.1)$$

$$\frac{dz_2}{dx} = z_3 = f_2(z_1, z_2, \dots, x) \quad (4.2)$$

⋮

$$\frac{dz_n}{dx} = \frac{1}{a_n}(-a_{n-1}z_n \dots - a_1z_2 - a_0z_1 + f(x)) \quad (4.n)$$

Each of the n first order ordinary differential equations are accompanied by one initial condition. These first order ordinary differential equations are simultaneous in nature but can be solved by the methods used for solving first order ordinary differential equations that we have already learned.

Example 1

Rewrite the following differential equation as a set of first order differential equations.

$$3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x}, \quad y(0) = 5, \quad y'(0) = 7$$

Solution

The ordinary differential equation would be rewritten as follows. Assume

$$\frac{dy}{dx} = z,$$

Then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

Substituting this in the given second order ordinary differential equation gives

$$3\frac{dz}{dx} + 2z + 5y = e^{-x}$$

$$\frac{dz}{dx} = \frac{1}{3}(e^{-x} - 2z - 5y)$$

The set of two simultaneous first order ordinary differential equations complete with the initial conditions then is

$$\frac{dy}{dx} = z, \quad y(0) = 5$$

$$\frac{dz}{dx} = \frac{1}{3}(e^{-x} - 2z - 5y), \quad z(0) = 7.$$

Now one can apply any of the numerical methods used for solving first order ordinary differential equations.

Example 2

Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 2, \quad \text{find by Euler's method}$$

a) $y(0.75)$

b) the absolute relative true error for part(a), if $y(0.75)|_{exact} = 1.668$

c) $\frac{dy}{dt}(0.75)$

Use a step size of $h = 0.25$.

Solution

First, the second order differential equation is written as two simultaneous first-order differential equations as follows. Assume

$$\frac{dy}{dt} = z$$

then

$$\frac{dz}{dt} + 2z + y = e^{-t}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y$$

So the two simultaneous first order differential equations are

$$\frac{dy}{dt} = z = f_1(t, y, z), \quad y(0) = 1 \tag{E2.1}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y = f_2(t, y, z), \quad z(0) = 2 \tag{E2.2}$$