# The Inverse Laplace Transform



## Inverse Laplace Transform by Partial Fraction Expansion

This technique uses Partial Fraction Expansion to split up a complicated fraction into forms that are in the Laplace Transform table. As you read through this section, you may find it helpful to refer to the section on partial fraction expansion techniques.

### Distinct Real Roots

Consider first an example with distinct real roots.

$$F(s) = \frac{s+1}{s(s+2)} = \frac{A_1}{s} + \frac{A_2}{s+2}$$

We can find the two unknown coefficients using the "cover-up" method.

$$A_1 = \left.\frac{s+1}{s(s+2)}\right|_{s=0} = \frac{1}{2}$$

$$A_2 = \left.\frac{s+1}{s(s+2)}\right|_{s=-2} = \frac{-1}{-2} = \frac{1}{2}$$

So

$$F(s) = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s+2}$$

and

$$f(t) = \frac{1}{2}U(t) + \frac{1}{2}e^{-2t}U(t)$$

(where U(t) is the unit step function) or expressed another way

$$f(t)=\frac{1}{2}+\frac{1}{2}e^{-2t}, \quad t>0$$

The last two expressions are somewhat cumbersome. Unless there is confusion about the result, we will assume that all of our results are implicitly 0 for t<0, and we will write the result as

$$f(t)=\frac{1}{2}+\frac{1}{2}e^{-2t}$$

## Repeated Real Roots

Consider next an example with repeated real roots (in this case at the origin, s=0).

$$F(s)=\frac{s^2+1}{s^2(s+2)}=\frac{A_1}{s+2}+\frac{A_2}{s}+\frac{A_3}{s^2}$$

We can find two of the unknown coefficients using the "cover-up" method.

$$A_1=\left.\frac{s^2+1}{s^2\cancel{(s+2)}}\right|_{s=-2}=\frac{5}{4}$$

$$A_3=\left.\frac{s^2+1}{\cancel{s^2}(s+2)}\right|_{s=0}=\frac{1}{2}$$

We find the other term using cross-multiplication:

$$s^2+1=s^2(s+2)\left(\frac{A_1}{s+2}+\frac{A_2}{s}+\frac{A_3}{s^2}\right)$$

$$=s^2A_1+s(s+2)A_2+(s+2)A_3$$

Equating like powers of "s" gives us:

| power of "s" | Equation |
|---|---|
| $s^2$ | $1=A_1+A_2$ |
| $s^1$ | $0=2A_2+A_3$ |
| $s^0$ | $1=2A_3$ |

We could have used these relationships to determine $A_1$, $A_2$, and $A_3$. But $A_1$ and $A_3$ were easily found using the "cover-up" method. The top relationship tells us that $A_2$=-0.25, so

$$F(s)=\frac{5}{4}\frac{1}{s+2}-\frac{1}{4}\frac{1}{s}+\frac{1}{2}\frac{1}{s^2}$$

and

$$f(t) = \frac{5}{4}e^{-2t} - \frac{1}{4} + \frac{1}{2}t$$

(where, again, it is implicit that f(t)=0 when t<0).