Linear Open Loop Systems

1st Order Systems

Output modeled with a 1st order ODE:

$$a_1 \frac{dy}{dt} + a_0 y = b \cdot f(t).$$

If $a_0 \neq 0$, then:

$$\frac{a_1}{a_0}\frac{dy}{dt} + y = \frac{b}{a_0} f(t) \quad \Rightarrow \quad \tau_p \frac{dy}{dt} + y = K_p \cdot f(t)$$

where: τ_p is the *time constant*.
K_p is the *steady state gain*, *static gain*, or *gain*.

For deviation variables, where $y(0) = f(0) = 0$, the Laplace transform will be:

$$(\tau_p s + 1)\bar{y}(s) = K_p \bar{f}(s) \quad \Rightarrow \quad G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau_p s + 1}$$

This transfer function is referred to as *1st order lag*, *linear lag*, or *exponential transfer lag*.

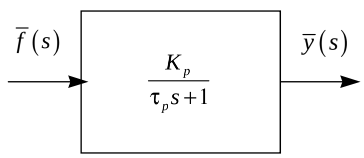
But what if $a_0 = 0$? Then:

$$a_1 \frac{dy}{dt} = bf(t) \quad \Rightarrow \quad \frac{dy}{dt} = \frac{b}{a_1} f(t) \quad \Rightarrow \quad \frac{dy}{dt} = K_p' \cdot f(t).$$

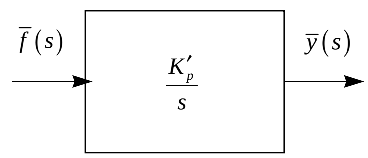
For deviation variables, the Laplace transform will be:

$$s\bar{y}(s) = K_p' \cdot \bar{f}(s) \quad \Rightarrow \quad G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p'}{s}.$$

This transfer function is referred to as *purely capacitive* or *pure integrator*.



1st Order Lag



Pure Integrator

Example 1st Order Systems — Mercury Thermometer

Last time we developed the following equation for the reading from a mercury thermometer:

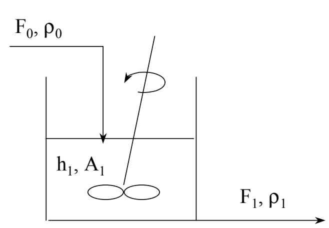
$$\frac{m\hat{C}_p}{hA}\frac{dT}{dt} = T_a - T \quad \Rightarrow \quad \frac{m\hat{C}_p}{hA}\frac{dT}{dt} + T = T_a$$

So this is a 1st order lag system with:

$$\tau_p = \frac{m\hat{C}_p}{hA}$$
$$K_p = 1$$

Example 1st Order Systems — Mass Storage in Tank

Mass storage in a tank is a 1st order system, but we don't know which type until we say something about how the flow out of the tank is controlled.



For constant density & constant cross-sectional area:

$$\frac{d(\rho_1 Ah)}{dt} = \rho_0 F_0 - \rho_1 F_1 \quad \Rightarrow \quad A\frac{dh}{dt} = F_0 - F_1 .$$

For flow through a valve where we can linearize to $F_1 \approx C_v h = h/R$, then:

$$A\frac{dh}{dt} = F_0 - \frac{h}{R} \quad \Rightarrow \quad AR\frac{dh}{dt} + h = RF_0 .$$

So this is a 1st order lag system with:

$$\tau_p = AR$$
$$K_p = R$$

However, if the flowrate out is controlled separately from the level in the tank, e.g., with a pump, then:

$$A\frac{dh}{dt} = F_0 - F_1 \quad \Rightarrow \quad \frac{dh}{dt} = \frac{F_0 - F_1}{A} .$$

So this is pure integrator system with:

$$f(t) = F_0 - F_1$$
$$K_p' = \frac{1}{A} .$$

Response of 1st Order Systems

Look at response to 4 typical driving functions.

Impulse disturbance. $f(t) = \alpha \cdot \delta(0) \quad \Rightarrow \quad \bar{f}(s) = \alpha$. So, if 1st order lag:

$$\bar{y}(s) = G(s)\cdot \bar{f}(s) = \frac{K_p}{\tau_p s + 1}\alpha = \frac{\frac{\alpha K_p}{\tau_p}}{s + \frac{1}{\tau_p}} \quad \Rightarrow \quad y(t) = \alpha \frac{K_p}{\tau_p} e^{-t/\tau_p}$$

If pure integrator:

$$\bar{y}(s) = G(s)\cdot \bar{f}(s) = \frac{K_p'}{s}\alpha \quad \Rightarrow \quad y(t) = \alpha K_p'$$

Unit step change. $f(t) = \alpha \cdot H(t) \quad \Rightarrow \quad \bar{f}(s) = \alpha / s$. So, if 1st order lag:

$$\bar{y}(s) = G(s) \cdot \bar{f}(s) = \frac{K_p}{\tau_p s + 1} \frac{\alpha}{s} = \alpha K_p \left(\frac{1}{s} - \frac{\tau_p}{\tau_p s + 1} \right)$$

$$= \alpha K_p \left(\frac{1}{s} - \frac{1}{s + \frac{1}{\tau_p}} \right)$$

$$\therefore \quad y(t) = \alpha K_p \left(1 - e^{-t/\tau_p} \right).$$

Notice that K_p is the fraction of the value of the input disturbance that will show up on the output signal. Also notice that the slope is:

$$\frac{dy}{dt} = \frac{\alpha K_p}{\tau_p} e^{-t/\tau_p} \quad \Rightarrow \quad \left. \frac{dy}{dt} \right|_{t=0} = \frac{\alpha K_p}{\tau_p}$$

If the system would maintain its initial rate of change, then it would achieve it ultimate value in one time constant, i.e., when $t = \tau_p$. In reality, the final value is reached in an exponential decay manner — in reality, it takes about $4\tau_p$ to reach the ultimate value (when $y \approx 0.98\alpha K_p$).